Math 512B. Homework 3. Solutions

The symbol $\lim_{x\to\infty} f(x)$ means " the limit of $f(x)$ as x approaches ∞." We say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is a number M such that, for all x,

$$\text{if } x > M, \text{ then } |f(x) - L| < \varepsilon.$$

A similar definition applies to $\lim_{x\to-\infty} f(x) = L$.

Problem 1. The limit $\lim_{N\to\infty} \int_a^N f$, if it exists, is denoted by $\int_a^\infty f$ (or by $\int_a^\infty f(x)\cdot dx$), and called an "improper integral."

(i) Find $\int_1^\infty x^r \cdot dx$ if $r < -1$.

(ii) Prove that $\int_1^\infty \frac{1}{x} \cdot dx$ does not exits

(iii) Does $\int_0^\infty \frac{1}{1+x^2} \cdot dx$ exist?

The improper integral $\int_{-\infty}^a f$ is defined as $\lim_{N\to-\infty} \int_N^a f$, as expected, but another kind of improper integral $\int_{-\infty}^\infty f$ is defined as $\int_0^\infty f + \int_{-\infty}^0 f$, provided both improper integrals exist.

(iv) Prove that $\int_{-\infty}^\infty \frac{1}{1+x^2} \cdot dx$ exist.

(v) Prove that $\lim_{N\to\infty} \int_{-N}^N x \cdot dx$ exists, but the improper integral $\int_{-\infty}^\infty x \cdot dx$ does not exist.

(vi) (Not required) Prove that the improper integral $\int_\pi^\infty \frac{\sin x}{x} \cdot dx$ exists, but $\int_\pi^\infty \frac{|\sin x|}{x} \cdot dx$ does not exist.

Solution. (i) $\int_1^N x^r dx = \frac{N^{r+1}-1}{r+1}$. If $r < -1$, then $\lim_{N\to\infty} N^{r+1} = 0$ and so $\int_1^\infty x^r dr = \frac{-1}{1+r}$

(ii) The integral $\int_1^N \frac{1}{x} dx = \log N$ and $\lim_{N\to\infty} \log N$ does not exist.

(iii) Yes. $\int_0^N \frac{1}{1+x^2}\, dx = \arctan N - \arctan 0$ and $\lim_{N\to\infty} (\arctan N - \arctan 0) = \frac{\pi}{2}$.

(iv) Use (iii) and the fact that $\int_{-N}^0 \frac{1}{1+x^2}\, dx = \int_0^N \frac{1}{1+x^2}\, dx$ to obtain

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx = \pi.$$

(v) $\int_{-N}^N x\, dx = 0$ for all N, but $\lim_{N\to\infty} \int_0^N x dx = \lim_{N\to\infty} \frac{N^2}{2}$ does not exist.

Problem 2. There is another kind of improper integral in which the interval is bounded but the function is unbounded.

(i) If $a > 0$ and $-1 < r < 0$, find $\lim_{\varepsilon\to 0+} \int_\varepsilon^a x^r \cdot dx$. This limit is denoted $\int_0^a x^r \cdot dx$, even though the function $f(x) = x^r$ is not bounded on $[0, a]$ (for $-1 < r < 0$), no matter how we define $f(0)$.

(ii) Suppose that f is continuous on $[0, 1]$. Find

$$\lim_{x\to 0+} x \int_x^1 \frac{f(t)}{t} \cdot dt.$$

(iii) (Not required.) The integral $\int_0^\infty \frac{1}{x^2+\sqrt{x}} \cdot dx$ does not fall into any of the two kind of improper integrals previously described in these problems. Can you give it a meaning? (Break up the interval $(0,\infty)$ at 1.)

Solution. (i) Since $r \neq -1$, $\int_\varepsilon^a x^r \cdot dx = \frac{a^{r+1} - \varepsilon^{r+1}}{r+1}$, and so $\lim_{\varepsilon\to 0+} \frac{a^{r+1} - \varepsilon^{r+1}}{r+1} = \frac{a^{r+1}}{r+1}$ because $0 < 1 + r < 1$.

(ii) Because f is continuous on $[0, 1]$ there is a constant C such that $|f(x)| \le C$ for all x in $[0, 1]$. Thus, if $0 < x < 1$, then

$$\left| x \int_x^1 \frac{f(t)}{t}\, dt \right| \le Cx \int_x^1 \frac{1}{t}\, dt = -Cx \log x.$$

Since $\lim_{x\to 0+} x \log x = 0$, we obtain

$$\lim_{x\to 0+} x \int_x^1 \frac{f(t)}{t}\, dt = 0.$$

Problem 3. Determine if the following sequences of functions converge pointwise or uniformly on the given interval. In case that there is pointwise convergence, then you must also identify the limit function.

(i) $f_n(x) = \frac{\sin nx}{n}$, $0 \le x \le 1$.

(ii) $g_n(x) = \frac{1}{n} \exp(-nx)$, $0 \le x < \infty$.

(iii) $h_n(x) = nx(1 - x^2)^n$, $0 \le x \le 1$.

Solution. In this problem, the symbol $\mathbf{0}$ denotes the function which is constant equal to 0; that is $\mathbf{0}(x) = 0$ for all x.

(i) We have $\frac{|\sin nx|}{n} \le \frac{1}{n}$ for all x and n, so $\lim_{n\to\infty} f_n(x) = 0$. Thus f_n converges uniformly to the function $\mathbf{0}$ on $[0, 1]$.

(ii) The sequence $g_n \to \mathbf{0}$ uniformly on $[0,\infty)$ because $\left|\frac{e^{-nx}}{n}\right| \le \frac{1}{n}$ for all $x \ge 0$.

(iii) We know that if $0 < a < 1$, then $\lim_{n\to\infty} na^n = 0$. Therefore, $\lim_{n\to\infty} h_n(x) = 0$ for all x in $[0, 1]$ (clearly, $h_n(0) = h_1(1) = 0$ for all n).

The convergence $h_n \to \mathbf{0}$ is not uniform. Indeed, $\int_0^1 h_n = \frac{-n}{2(n+1)}$ converges to $-1/2$ as $n \to \infty$, but $\int_0^1 \mathbf{0} = 0$. That is,

$$\lim_{n\to\infty} \int_0^1 h_n \neq \int_0^1 \mathbf{0} = 0$$

If the convergence was uniform, the limit of the integrals would equal the integral of the limit.

Problem 4. Suppose that $\{f_n\}$ is a sequence of functions which converges uniformly to f on the interval $[a, b]$. Prove that if each f_n is integrable on $[a, b]$, then the limit function f is also integrable on $[a, b]$.

Solution. (a) f is bounded. Because uniform convergence $f_n \to f$, for $\varepsilon = 1$ there is N such that $|f_N(x) - f(x)| \leq 1$ for all x in $[a, b]$. Because f_N is integrable on $[a, b]$, it is bounded. Thus there is M such that $|f_N(x)| \leq M$ for all x in $[a, b]$. Therefore,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M$$

proving that f is also bounded.

(b) f is integrable. Let $\varepsilon > 0$. Because $f_n \to f$ uniformly, there is N such that $|f_N(x) - f(x)| < \varepsilon/3$. Because f_N is integrable, there is a partition P of $[a, b]$ for which the lower and upper sums of f satisfy:

$$(0 \leq)\, U(f_N, P) - L(f_N, P) < \varepsilon/3.$$

For any interval $[t_{i-1}, t_i]$ of the partition P we have $|m_i(f) - m_i(f_N)| \leq \varepsilon/3$ and $|M_i(f) - M_i(f_N)| \leq \varepsilon/3$ because $|f(x) - f_N(x)| < \varepsilon/3$. Therefore, $|U(f, P) - U(f_N, P)| \leq \varepsilon/3$ and $|L(f, P) - L(f_N, P)| < \varepsilon/3$, and

$$\begin{aligned} |U(f,P) - L(f,P)| &\leq |U(f,P) - U(f_N,P)| \\ &\quad + |U(f_N,P) - L(f_N,P)| \\ &\quad + |L(f_N,P) - L(f,P)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon, \end{aligned}$$

as desired.