Problem 1. Prove that if X is contractible and $A \subset X$ is a retract of X, then A is contractible.

Solution. By definition of contractible, the identity mapping $\mathrm{id}_X : X \to X$ is homotopic to a constant mapping $c_{x_0} : X \to \{x_0\}$, equivalently, there is a mapping $H : X \times [0,1] \to X$ such that $H(x,0) = x$ for all x in X and $H(x,1) = x_0$ for all x in X.
Let $r : X \to A$ be the retraction mapping, let $a_0 = r(x_0)$, and let $K : A \times [0,1] \to A$ be the composition $r \circ H$. This mapping K is a homotopy of id_A to the constant mapping c_{a_0} because it is continuous and satisfies $K(a,0) = r(a) = a$ for all a in A and $K(a,1) = r(x_0) = a_0$ for all a in A. Thus A is contractible. □

Problem 2. Prove that if $A \subset X$ is a deformation retract, then A and X are homotopically equivalent.

Solution. If A is a deformation retraction, then there is a retraction $r : X \to A$ such that $j \circ r : X \to X$ is homotopic to the identity of X, where $j : A \to X$ is the inclusion. Let $H : j \circ r \simeq \mathrm{id}_X$ be this homotopy.
The homotopy equivalence $X \simeq A$ is realized by the mappings $r : X \to A$ and $j : A \to X$, because $r \circ j = \mathrm{id}_A$ and by the above, $H : j \circ r \simeq \mathrm{id}_X$. □