Problem 1. Let X be a set and $A \subset X$ a subset. Let $\mathcal{T}$ be the collection consisting of $\emptyset$ and of all subsets of X that contain A.

(a) Prove that $\mathcal{T}$ is a topology on X.

(b) Find the interior and the closure of A in that topology $\mathcal{T}$.

Solution. (a) $\mathcal{T}$ is a topology on X.

(a.i) $\emptyset \in \mathcal{T}$ by definition

(a.ii) $X \in \mathcal{T}$ because $A \subset X$

(a.iii) If U and V are in $\mathcal{T}$, and if (1) $A \subset U$, and (2) $A \subset V$ are both true, then $A \subset U \cap V$, implying that $U \cap V \in \mathcal{T}$. If at least one of (1) or (2) is false, then either $U = \emptyset$, or $V = \emptyset$, or both, and thus $U \cap V = \emptyset$, which is also in $\mathcal{T}$.

(a.iv) Let $\Sigma \subset \mathcal{T}$ be an arbitrary family. If some element of Σ contains A, then $\bigcup_{U\in\Sigma} U$ also contains A; if not, then $\bigcup_{U\in\Sigma} U = \emptyset$. In either case, $\bigcup_{U\in\Sigma} U$ is in $\mathcal{U}$.

(b.i) Because $A \subset A$, the set A is open. Hence the interior $A^\circ = A$.

(b.ii) The closure of A is the intersection of all closed sets that contain A. Let F be a closed set such that $A \subset F$. Then $X \setminus F$ is an open set that satisfies $(X \setminus F) \cap A = \emptyset$.

If $A \neq \emptyset$, then $X \setminus F = \emptyset$, because any non-empty open set must contain A, and so $F = X$. Therefore, if $A \neq \emptyset$, then $A^- = X$.

If $A = \emptyset$, then $A^- = A = \emptyset$.

□