Problem 1. (a) Let X be simply connected. Prove that if $A \subset X$ is a retract, then A is also simply connected.

(b) Give an example of a non-simply connected space X containing a simply connected retract $A \subset X$.

Solution. A space is simply connected if it is path connected and its fundamental group (any base point) is the trivial group.

(a) Let $r : X \to A$ be the retraction mapping. If a, b are two points in A, then there is a path $\alpha : [0, 1] \to X$ from a to b. The composition $r \circ \alpha : [0, 1] \to A$ is a path in A from a to b. Therefore A is path connected.

Let a be any point in A and let α be a loop at a in A. Then α is a loop at a in X. Since $\pi_1(X; a)$ is trivial, the loop α is homotopic to the constant loop c_a relative to endpoints. If $H : [0, 1] \times [0, 1] \to X$ is a homotopy in X between α and c_a relative to endpoints, then the composition $r \circ H : [0, 1] \times [0, 1] \to A$ is a homotopy of α and c_a in A relative to endpoints. Therefore, $\pi_1 A; a)$ is the trivial group.

(b) If $a \in X$, then $A = \{a\}$ is a retract of X that is simply connected because the only loops on $\{a\}$ are the constant path c_a.

□

Problem 2. Let $X = \mathbf{N} \cup \{\infty\}$, where $\mathbf{N}$ is the set of natural numbers, endowed with the topology in which $F \subset X$ is closed if and only if either $F = X$ or else F is a finite subset of $\mathbf{N}$.

(a) Prove that X is path connected.

(b) Prove that $\pi_1(X; \infty)$ is the trivial group.

Solution. (a) Let m, n be any points in X. The mapping $\alpha : [0, 1] \to X$ given by $\alpha(0) = n$, $\alpha(1) = m$, and $\alpha(s) = \infty$ for $0 < s < 1$ is continuous. Indeed, if F is closed in X, then

$$\alpha^{-1}F = \begin{cases} \{0, 1\} & \text{if } n, m \in F, \\ \{0\} & \text{if } n \in F \text{ and } m \notin F, \\ \{1\} & \text{if } n \notin F \text{ and } m \in F, \\ \emptyset & \text{if } n \notin F \text{ and } m \notin F, \end{cases}$$

all of which are closed subsets of $[0, 1]$

(b) Let $\alpha, \beta : [0, 1] \to X$ be two loops based at ∞, that is, two paths such that $\alpha(0) = \beta(0) = \alpha(1) = \beta(1) = \infty$. Let $H : [0, 1] \times [0, 1] \to X$ be given by

$$H(s, t) = \begin{cases} \alpha(s), & \text{if } t = 0 \\ \beta(s), & \text{if } t = 1 \\ \infty, & \text{if } 0 < t < 1 \end{cases}$$

Then H is a homotopy from α to β relative to endpoints. Indeed, H has the correct boundary values, namely, $H(s, 0) = \alpha(s)$, $H(s, 1) = \beta(s)$ for all $0 \leq s \leq 1$, and $H(0, t) = H(1, t) = \infty$ for all $0 \leq t \leq 1$, and H is also continuous because if $F \subset X$ is closed and $F \neq X$, then $H^{-1}F = \alpha^{-1}F \cup \beta^{-1}F$ is also closed in $[0, 1] \times [0, 1]$.

□