

Homework 12

¶ 1. Let $A \subset X$ be a retract, let $r : X \rightarrow A$ be the retraction mapping, and let $j : A \rightarrow X$ be the inclusion mapping. Prove that, for any a in A , the homomorphism $r_* : \pi_1(X; a) \rightarrow \pi_1(A; a)$ is surjective, and the homomorphism $j_* : \pi_1(A; a) \rightarrow \pi_1(X; a)$ is injective.

¶ 2. Prove that the unit circle $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ is not a retract of the plane \mathbf{R}^2 .

¶ 3. Prove that for a path connected space X the following are equivalent:

- (a) X is simply connected;
- (b) every continuous mapping $f : S^1 \rightarrow X$ is homotopic to a constant mapping.
- (c) every continuous mapping $f : S^1 \rightarrow X$ admits a continuous extension to the closed disk B^2 ;
- (d) any two paths $\alpha, \beta : [0, 1] \rightarrow X$ with the same initial and final points are homotopic relative to endpoints.

¶ 4. If X is a topological space and γ is a path in X from $\gamma(0) = a$ to $\gamma(1) = b$, then the mapping $\gamma_* : \pi_1(X; a) \rightarrow \pi_1(X; b)$ given by $\gamma_*[\alpha] = [\gamma^{-1}\alpha\gamma]$ is an isomorphism.

Prove that $\gamma_* = \gamma'_*$ for paths $\gamma, \gamma' \in \Pi(X; a, b)$ if and only if $\pi_1(X, a)$ is commutative.

¶ 5. Let (X, a) and (Y, b) be pointed spaces. Prove that the fundamental group $\pi_1(X \times Y; (a, b))$ is isomorphic to the direct product $\pi_1(X; a) \times \pi_1(Y; b)$.

¶ 6. Prove that if there are two simply connected open subsets U, V of X such that $X = U \cup V$ and $U \cap V$ is non-empty and simply connected, then X is simply connected.

¶ 7. Suppose that (X, a) and (Y, b) are path connected topological spaces and $f : (X, a) \rightarrow (Y, b)$ is a continuous mapping, and let $f_* : \pi_1(X; a) \rightarrow \pi_1(Y; b)$ be the induced homomorphism. Prove or give a counterexample.

- (a) If f is injective, then f_* is injective.
- (b) If f is surjective, then f_* is surjective.
- (c) If f is bijective, then f_* is an isomorphism.

¶ 8. Let S^n be the unit sphere in \mathbf{R}^{n+1} , and let $a = (0, 0, \dots, 0, 1)$ be the north pole. For $n \geq 2$, prove the following.

- (a) Any path $\alpha \in \Pi(S^n; a, a)$ is homotopic relative to endpoints to a path β whose image does not fill the whole sphere, that is, such that $\beta([0, 1]) \not\subset S^n$.
- (b) Any path $\alpha \in \Pi(S^n; a, a)$ such that $\alpha([0, 1]) \not\subset S^n$ is homotopic relative to endpoints to the constant path c_a .
- (c) S^n is simply connected.

¶ 9. Prove that $X = \mathbf{R}^n \setminus \{0\}$ is simply connected if $n \geq 3$.

¶ 10. Let M be the space obtained as the quotient space of \mathbf{R}^2 modulo the equivalence relation given by $(x, y) \sim (x', y')$ if and only if $x - x' = m$ is an integer and $y - (-1)^m y' = 0$. Prove that $\pi_1(M, a)$ is infinite cyclic.

¶ 11. Let $X = \mathbf{N} \cup \{\infty\}$, where \mathbf{N} is the set of natural numbers, endowed with the topology in which $F \subset X$ is closed if and only if either $F = X$ or else F is a finite subset of \mathbf{N} . Prove that X is simply connected.