Homework 10 Solutions

¶ 1. Prove that a Hausdorff space is locally compact if and only if every point has a compact neighborhood.

Solution. Let K be a compact neighborhood of x. Let U be any neighborhood of x. Then $V = (K \cap U)^\circ$ is an open neighborhood of x. The space V^- is compact and Hausdorff, and V is a neighborhood of x in V^-. Therefore, cf. Problem 5 below) there is an open neighborhood W of x in V^- such that $\mathrm{Cl}_{V^-} W \subset V$. Since V is open in X, W is open in X. The set $\mathrm{Cl}_{V^-} W$ is closed in V^-, hence compact. It follows that $\mathrm{Cl}_{V^-} W$ is a compact neighborhood of x contained in U, and therefore x has a neighborhood base formed by compact sets. □

¶ 2. (a) In a locally compact Hausdorff space, the intersection of an open set with a closed set is locally compact.

(b) A locally compact subset of a Hausdorff space is the intersection of a closed set and an open one.

(c) A dense subset of a compact Hausdorff space is locally compact if and only if it is open.

Solution. (a) Suppose X is locally compact and Hausdorff. If U is open in X and $x \in U$, then there is a compact neighborhood K of x contained in U. Thus U is also locally compact. If F is closed in X and $x \in F$, then x has a compact neighborhood K in X. But $K \cap F$ is a compact neighborhood of x in F, so it is locally compact. Since the intersection of two locally compact spaces in X is locally compact, the intersection of a closed and a open in X is locally compact.

□

¶ 3. (a) Prove that if X is locally compact and $f : X \to Y$ is continuous, open and onto, then Y is also locally compact.

(b) Prove that a non-empty product of of finitely many locally compact spaces is locally compact. (In general, a non-empty product is locally compact spaces if and only if all its factors are locally compact, and all but finitely many of them are compact.)

¶ 4. Let X be Hausdorff and locally compact, and view X as a subspace of its one-point compactification X^*. Let $f : X \to \mathbf{R}$ be continuous. Prove that f admits a continuous extension to X^* (that is, there is $F : X^* \to \mathbf{R}$ continuous such that $F(x) = f(x)$ for all x in X) if and only if for each $\epsilon > 0$ there is a compact subset K_ϵ of X such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in X \setminus K_\epsilon$.

Solution. Let ∞ denote the point in $X^* \setminus X$. If $F : X^* \to \mathbf{R}$ is continuous, then it is continuous at ∞. Thus, given any $\epsilon > 0$ there is an open neighbored $U \subset X^*$ of ∞ such that if $x \in U$, then $|F(x) - F(\infty)| < \epsilon/2$. Because U is open in X^* and contains ∞, the set $K = X \setminus U$ is compact. If x, y are not in K, then x, y are in U and $|F(x) - F(y)| \le |F(x) - F(\infty)| + |F(\infty) - F(y)| < \epsilon$.
Conversely, if $f : X \to \mathbf{R}$ has the stated property, then for each $\epsilon = 1, 1/2, 1/3, \cdots$, there is a compact set $K_{1/n}$ of X such that $|f(x) - f(y)| < 1/n$ for all x, y in $X \setminus K_{1/n}$. The set $U_n = X^* \setminus (K_1 \cup K_{1/2} \cup K_{1/n})$ is an open neighborhood of ∞ (the union of finitely many compact sets is a compact set), and by the above $|f(x) - f(y)| < 1/n$ if x, y are in $U_n \cap X$. Therefore, $f(X \cap U)$ is a bounded set in $\mathbf{R}$, and so its closure $H_n = f(X \cap U_n)$ is compact. We have constructed a decreasing sequence $H_n \supset H_{n+1}$ of non-empty compact sets with diameter $\operatorname{diam} H_n \to 0$. Therefore, there is a unique point $a \in \bigcap H_n$.
Define $F : X^* \to \mathbf{R}$ by $F(x) = f(x)$ is x is in X and $F(\infty) = a$.
We prove that F is continuous at each point. Suppose that $x \in X^*$ is $x \neq \infty$. Because X is Hausdorff and locally compact, X^* is Hausdorff. Therefore, there are disjoint open sets U and V in X^*, with x in V and ∞ in U. On V, $F = f$, and so F is continuous on V, hence at x because V is open in X^*.
We now prove that F is continuous at ∞. Given $\epsilon > 0$, let n be such that $1/n < \epsilon$. Then U_n is an open neighborhood of ∞ that satisfies $F(U_n) \subset H_n$. By the above, H_n contains $F(\infty) = a$ and has diameter $\le 1/n$. Hence $F(U_n) \subset (F(\infty) - \epsilon, F(\infty) + \epsilon))$. □

¶ 5. A space X is regular if given any closed $F \subset X$ and any $x \in X \setminus F$, there are open disjoint subsets of X, one containing F and the other containing x. A space that is regular and T_1 is said to be a T_3 space (or to satisfy the T_3-axiom).
Prove that the following properties are equivalent for a topological space X.

(a) X is regular.

(b) If U is open in X and $x \in U$, there exists an open set V such that $x \in V \subset V^- \subset U$.

(c) each point has a neighborhood base consisting of closed sets.

Solution. (a) $\Rightarrow$ (b) $X \setminus U$ is a closed set not containing x.

(b) $\Rightarrow$ (c) Because of (b), for every x and every neighborhood U of x there is a neighborhood V with $V^- \subset U$, such V^- is a closed neighborhood.

(c) $\Rightarrow$ (a) If F is a closed set not containing x, then $X \setminus F$ is a neighborhood of x, so there is a closed neighborhood A of x with $A \subset X \setminus F$. Then A° and $X \setminus A$ are disjoint open sets, the first containing x, the last containing F. and F.

□

¶ 6. A space X is completely regular (or Tichonoff) if whenever F is a closed subset of X and $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f|F = 0$. Sometimes it may be more convenient to use the following equivalent definition: given $x \in X$ and a neighborhood U of x, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus U) = 1$. A space that is completely regular and T_1 is said to be a $T_{3\frac{1}{2}}$-space (or to satisfy the $T_{3\frac{1}{2}}$-axiom).

(a) Prove that a subspace of a $T_{3\frac{1}{2}}$-space (Tichonoff space) is a $T_{3\frac{1}{2}}$-space (Tichonoff space).

(b) A nonempty product space is $T_{3\frac{1}{2}}$ (Tichonoff) if and only if each factor is $T_{3\frac{1}{2}}$ (Tichonoff).

Solution. (a) Let X be $T_{3\frac{1}{2}}$ and $Y \subset X$. Let $x \in Y$ and F a closed subset of Y which does not contain x. Then $F = G \cap Y$ for some closed subset G of X. Because x is not in G and X is $T_{3\frac{1}{2}}$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f|G \equiv 0$. The restriction $f|Y$ of f to Y is continuous on Y and verifies that Y is $T_{3\frac{1}{2}}$.

(b) Suppose that each X_α is $T_{3\frac{1}{2}}$ and let $x \in X$ and $A \subset X$ closed not containing x. Then there is a basic neighborhood $\pi_1^{-1}U_1 \cap \cdots \cap \pi_n^{-1}U_n$ of x and disjoint from A. For each α_k, $k = 1, \cdots, n$, there is a continuous function $f_k : X_{\alpha_k} \to [0, 1]$ such that $f_k(x_{\alpha_k}) = 1$ and $f_k(X_{\alpha_k} \setminus U_{\alpha_k}) = 0$. Define $g : X \to [0, 1]$ by $g = \min f_k \circ \pi_{\alpha_k}$. Then f is continuous, $f(x) = 1$, and $f(A) = 0$.

□

¶ 7 (Moore plane)**.** Let M be the set of points $(x, y) \in \mathbf{R}^2$ with $y \geq 0$. Define a topology on M via neighborhoods as follows: A neighborhood of a point (x, y) in M with $y > 0$ is any subset of M that contains an open Euclidean disc centered at (x, y); a neighborhood of a point $(x, 0)$ is any subset of M that contains $(x, 0)$ and an open Euclidean disc contained in M that is tangent to $y = 0$ at $(x, 0)$.
Prove that M is $T_{3\frac{1}{2}}$ but it not T_4.

Proof. The Moore plane M is Tichonoff. Let $x \in M$ and U a base neighborhood of x. That is, U is an open disc centered at x if x is in the open upper half plane, or $U = V \cup \{x\}$, V a disc tangent at x in the other case. Define f to be 0 at x and 1 on $X \setminus U$, and then extending linearly along the line segments joining x to the points on the boundary of U.
The second part is difficult. Let $Y \subset M$ denote the x-axis. Its induced topology is discrete. The points (p, q), where q and q are rational and $q > 0$ constitute a countable dense subset D of M. This implies that there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ continuous functions on D. Since Y is discrete with cardinality $\mathfrak{c}$, there are at least $2^{\mathfrak{c}}$ continuous functions on Y. Since $\mathfrak{c} < 2^{\mathfrak{c}}$, not all of them can be extended to X. □

¶ 8. Let X be a normal space. Let $A \subset X$ be a closed subset and $f : A \to \mathbf{R}$ continuous with $|f(x)| \leq c$ for each x in A. Prove that there is a continuous function $F : X \to \mathbf{R}$ such that

(a) $|F(x)| \leq c/3$ for all x in X.

(b) $|f(x) - F(x)| \leq 2c/3$ for all x in A.

Hint. Apply Uryshon's Lemma to the sets $f^{-1}[-c, -c/3]$ and $f^{-1}[c/3, c]$. Such function F is called a $1/3$-approximate extension of f.

Solution. Because the sets $H = f^{-1}[-c, -\frac{1}{3}c]$ and $K = f^{-1}[\frac{1}{3}c, c]$ are disjoint and closed in A, they are closed in X, and by the Urysohn lemma there exists a continuous function $g : X \to [0, 1]$ such that $g|H = 0$ and $g|K = 1$. Then take $F(x) = \frac{2}{3}c(g(x) - \frac{1}{2})$. □

¶ 9 (Tietze's Extension Theorem)**.** Let X be a normal space. Let A be a closed subset of X and let $f : A \to [-1, 1]$ be a continuous function. Then there is $F : X \to [-1, 1]$ continuous such that $F(a) = f(a)$ for all a in A (F is called a continuous extension of f).
To prove this, let F_1 a $1/3$-approximate extension of f (as defined in Problem 8), and inductively let F_{n+1} be a $1/3$-approximate extension of $f - (F_1 + \cdots + F_n)$.

(a) Prove that $|f(x) - \sum_{i=1}^{n} F_i(x)| \leq (2/3)^n$ for all x in A.

(b) Prove that $|F_{n+1}(x)| \leq 2^n/3^{n+1}$ for all $x \in X$.

(c) Prove that the series $\sum_{n=1}^{\infty} F_n$ converges uniformly on X to a continuous function $F : X \to [-1, 1]$.

(d) Prove that F is a continuous extension of $f : A \to [-1, 1]$.

¶ 10. Tietze's extension theorem was originally proved before Urysohn's lemma. Nowadays one proves Tietze's extension theorem as a corollary to Urysohn's lemma (as is done with Problem 8 and Problem 9 above). The question arises: can you prove Urysohn's Lemma as a corollary to Tietze's extension theorem?

Problem. Let A and B be disjpint closed subsets of the normal space X. Then $A \cup B$ is also a closed subset and the mapping $f : A \cup B \to [0, 1]$ given by $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all b in B is well defined and continous on $A \cup B$ because A and B are disjpint and closed. By Tietze extension theorem, there is a continuous mapping $F : X \to [0, 1]$ that extends f. This F satisfies $F(a) = 0$ for all a in A and $F(b) = 1$ for all b in B. □