¶ 1. A topological space is T_1 if given any two points, each of them lies in an open set not containing the other. Prove that the following are equivalent:

(a) X is T_1,

(b) Singletons (sets with exactly one point) are closed subsets.

(c) Any subset of X is the intersection of all the open sets containing it.

¶ 2. True or false:

(a) The union of finitely many compact subsets of a space is a compact set.

(b) The intersections of two compact subsets of a space is compact.

¶ 3. Let X be a compact, Hausdorff space.

(a) Prove that if K is closed in X and x is in $X \setminus K$, then there are open disjoint subsets of X, one containing K, the other containing x.

(b) Prove that if $U \subset X$ is an open neighborhood of x, then there is an open set V such that $x \in V \subset V^- \subset U$.

¶ 4. Let X be Hausdorff and Y compact and Hausdorff. Prove that $f : X \to Y$ is continuous if and only if the graph $\{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.

¶ 5. Let $A \times B$ be a compact subset of a product space $X \times Y$. Prove that if $W \subset X \times Y$ is an open subset such that $A \times B \subset W$, then there are open sets $U \subset X$ and $V \subset Y$ such that $A \times B \subset U \times V$ and $U \times V \subset W$.

¶ 6. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a countable family of compact subsets of a Hausdorff space X, such that $A_{n+1} \subset A_n$ for $n = 1, 2, \cdots$. Prove that if U is an open subset of X such that $\bigcap_{n=1}^{\infty} A_n \subset U$, then there is N such that $A_n \subset U$ for all $n > N$.

¶ 7. A space is **sequentially compact** if every sequence in the space has a convergent subsequence. A space is **countably compact** if every countable open covering has a finite subcovering.

(a) Prove that any sequentially compact space is countably compact.

(b) Prove that any sequentially compact metric space is compact.

¶ 8. A space is **second countable** it it has a countable base for its topology. A space is **Lindelöf** if every open cover has a countable subcovering.

(a) Prove that any second countable space is Lindelöf.

(b) Prove that a metric space is Lindelöf if and only if it is is second countable.

¶ 9. Let ℓ^2 denote Hilbert space, that is, the space of square summable sequences of real numbers, $\ell^2 = \left\{ (x_n) \,\middle|\, \sum_{n=1}^{\infty} x_n^2 < \infty \right\}$. It is a metric space with the distance function D defined by

$$D((x_n), (y_n)) = \left(\sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{1/2}.$$

(a) Prove that the closed balls $B(\mathbf{0}, \epsilon) = \left\{ (x_n) \in \ell^2 \,\middle|\, \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2} \leq \epsilon \right\}$ are not compact, and deduce that ℓ^2 is not locally compact.

Hilbert space ℓ^2 can be viewed as a subset of the countable product $\mathbf{R}^{\mathbf{N}}$. But the topology that it inherits from $\mathbf{R}^{\mathbf{N}}$ and the topology as metric space are distinct: ℓ^2 with the metric topology is not homeomorphic to ℓ^2 with the subspace topology from $\mathbf{R}^{\mathbf{N}}$. On the other hand, ℓ^2 and $\mathbf{R}^{\mathbf{N}}$ are homeomorphic (R. D. Anderson, *Bull. Amer. Math. Soc.* **72** (1966), 515–519.) The Hilbert cube $\mathbf{I}^{\omega}$ is the subset of ℓ^2 consisting of sequences (x_n) such that $0 \leq x_n \leq 1/n$ for all $n = 1, 2, 3, \cdots$.

(b) Prove that the mapping $f : \mathbf{I}^{\omega} \to [0, 1]^{\mathbf{N}}$ defined by $f(x_1, x_2, \cdots) = (x_1, 2x_2, 3x_3, \cdots)$ is bijective, continuous, and open. (Here $\mathbf{I}^{\omega}$ has the subspace topology from ℓ^2, and $[0, 1]^{\mathbf{N}}$ has the product topology.)

(c) Prove, using (b) or otherwise, that the Hilbert cube is compact.

¶ 10. Let X be Hausdorff and locally compact, and view X as a subspace of its one-point compactification X^*. Let $f : X \to \mathbf{R}$ be continuous. Prove that f admits a continuous extension to X^* (that is, there is $F : X^* \to \mathbf{R}$ continuous such that $F(x) = f(x)$ for all x in X) if and only if for each $\epsilon > 0$ there is a compact subset K_ϵ of X such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in X \setminus K_\epsilon$.

¶ 11. Chapter 2, Section 6: 1, 2, 3, 4, 5, 6, 8, 9, 10