

Homework 9. Solutions

¶ 1. A topological space is T_1 if given any two points, each of them lies in an open set not containing the other. Prove that the following are equivalent:

- (a) X is T_1 ,
- (b) Singletons (sets with exactly one point) are closed subsets.
- (c) Any subset of X is the intersection of all the open sets containing it.

Solution. (a) \Rightarrow (b) Let $\{x\} \subset X$ be a singleton. For each $y \neq x$ there is an open set U_y that contains x but not y and an open set V_y that contains y but not x . Thus $\{x\}$ is contained in the closed set $X \setminus V_y$, and also that

$$\{x\} = \bigcap_{y \neq x} X \setminus V_y.$$

- (b) \Rightarrow (c) Any $A \subset X$ is contained in the intersection of all the open subsets that contain it. To show that the reverse containment holds true, note that by (b), if x is not in A , then $X \setminus \{x\}$ is an open set that contains A but not x .
- (c) \Rightarrow (a) If $x \neq y$ in X , then $y \neq \{x\}$ and so, by (c), there is an open set that contains x but not y .

¶ 2. True or false:

- (a) The union of finitely many compact subsets of a space is a compact set.
- (b) The intersections of two compact subsets of a space is compact.

Solution. (a) True. If $A = H \cup K$ is the union of two compact subsets H, K of X , then any covering of A by open subsets of X will be a covering of H and of K .

- (b) False. Let X be the line with two origins, that is, the quotient of $\mathbf{R} \times \{0, 1\}$ by the equivalence relation that identifies $(x, 0)$ and $(y, 1)$ if $x = y = 0$. Let $p : \mathbf{R} \times \{0, 1\} \rightarrow X$ be the quotient mapping. The sets $p([0, 1] \times \{0\})$ and $p([0, 1] \times \{1\})$ are compact because they are continuous images of compact sets, but the intersection is homeomorphic to $(0, 1]$, which is not compact.

¶ 3. Let X be a compact, Hausdorff space.

- (a) Prove that if K is closed in X and x is in $X \setminus K$, then there are open disjoint subsets of X , one containing K , the other containing x .

- (b) Prove that if $U \subset X$ is an open neighborhood of x , then there is an open set V such that $x \in V \subset V^c \subset U$.

Solution. (a) Because X is Hausdorff and x is not in K , for each $y \in K$ there are disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. The collection $\{V_y \mid y \in K\}$ is an open covering of K . Because K is compact (it is a closed subset of the compact space X), there are finitely many y_1, y_2, \dots, y_n points in K such that $K \subset V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$. Let $V = V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$ and $U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$, so that U and V are open sets, with $K \subset V$ and $x \in U$. There are moreover disjoint because

$$\begin{aligned} U \cap V &= \left(\bigcap_{i=1}^n U_{y_i} \right) \cap \left(\bigcup_{j=1}^n V_{y_j} \right) \\ &= \bigcup_{j=1}^n \left(\bigcap_{i=1}^n (U_{y_i} \cap V_{y_j}) \right) \\ &\subset \bigcup_{j=1}^n (U_{y_i} \cap V_{y_j}) \\ &= \emptyset \end{aligned}$$

□

- ¶ 4. Let Y be compact and Hausdorff. Prove that $f : X \rightarrow Y$ is continuous if and only if the graph $\{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.

Solution. If Y is compact then the projection $X \times Y \rightarrow X$ is closed. Indeed, if A is a closed subset of $X \times Y$ and x is not in the projection of A onto X , then apply tube lemma to $\{x\} \times Y$ and the open $(X \times Y) \setminus A$ to find a W open nbhd of x in X such that $W \times Y$ is contained in $(X \times Y) \setminus A$. This shows that W does not intersect the projection of A onto X , and from this it follows that such projection is closed.

Assume that the graph, G , of $f : X \rightarrow Y$ is closed in $X \times Y$. If B is a closed subset of Y , then $\pi_Y^{-1}B = X \times B$ is a closed subset of $X \times Y$, and so $G \cap \pi_Y^{-1}B$ is also closed. Therefore, by the above, its image, $\pi_X(G \cap \pi_Y^{-1}B)$, via the projection $\pi_X : X \times Y \rightarrow X$ is closed in X . But $\pi_X(G \cap \pi_Y^{-1}B) = f^{-1}(B)$. Indeed, $x \in f^{-1}(B)$ if and only if $f(x) \in B$, if and only if $(x, f(x)) \in G \cap (X \times B)$.

Suppose that $f : X \rightarrow Y$ is continuous. If (x, y) is not in the graph, then $f(x) \neq y$. Because Y is Hausdorff, there are disjoint open set V and W in Y with y in V and $f(x)$ in W . Because f is continuous, $U = f^{-1}W$ is open in X . Then $U \times V$ is an open neighborhood of (x, y) that is disjoint from the graph G . Indeed, if $(z, w) \in (U \times V) \cap G$, then $w = f(z)$ is in V , and $f(z)$ is also in W , contradicting that V and W are disjoint. □

- ¶ 5. Let $A \times B$ be a compact subset of a product space $X \times Y$. Prove that if $W \subset X \times Y$ is an open subset such that $A \times B \subset W$,

then there are open sets $U \subset X$ and $V \subset Y$ such that $A \times B \subset U \times V$ and $U \times V \subset W$.

Solution. Let $x \in A$. We cover the compact set $\{x\} \times B$ by open boxes $U \times V \subset W$. Then there are finitely many of them, say $U_1 \times V_1, \dots, U_n \times V_n$, such that $\{x\} \times B \subset \bigcup_{i=1}^n U_i \times V_i \subset W$. Let

$$U_x = \bigcap_{i=1}^n U_i \text{ and } V_x = \bigcap_{i=1}^n V_i. \text{ Then } \{x\} \times B \subset U_x \times V_x \subset W.$$

The collection of sets $\{U_x \times V_x \mid x \in A\}$ is an open cover of the compact set $A \times B$. Let $U_1 \times V_1, \dots, U_n \times V_n$ be a finite subcover, and let $U = \bigcup_{i=1}^n U_i$ and $V = \bigcap_{i=1}^n V_i$. Then

$$A \times B \subset U \times V \subset W. \quad \square$$

¶ 6. Suppose that $\{A_n\}_{n=1}^\infty$ is a countable family of compact subsets of a Hausdorff space X , such that $A_{n+1} \subset A_n$ for $n = 1, 2, \dots$. Prove that if U is an open subset of X such that $\bigcap_{n=1}^\infty A_n \subset U$, then there is N such that $A_n \subset U$ for all $n > N$.

¶ 7. A space is **sequentially compact** if every sequence in the space has a convergent subsequence. A space is **countably compact** if every countable open covering has a finite subcovering.

- (a) Prove that any sequentially compact space is countably compact.
- (b) Prove that any sequentially compact metric space is compact.

Solution. (a) Suppose that X is sequentially compact and let $\mathcal{U} = \{U_1, U_2, \dots\}$ be a countable covering of X . If \mathcal{U} admits no finite subcovering, then for each $n = 1, 2, \dots$, there is x_n in $X \setminus (U_1 \cup \dots \cup U_n)$. Because X is sequentially compact, the sequence (x_n) has a subsequence (x_{n_k}) that converges to some point x_0 in X . Because \mathcal{U} is a covering, $x_0 \in U_m$ for some m . It then $x_{n_k} \in U_m$ for all k sufficiently large, in particular for some $n_k > m$. This contradicts that $x_{n_k} \in X \setminus (U_1 \cup \dots \cup U_{n_k})$. \square

¶ 8. A space is **second countable** if it has a countable base for its topology. A space is **Lindelöf** if every open cover has a countable subcovering.

- (a) Prove that any second countable space is Lindelöf.
- (b) Prove that a metric space is Lindelöf if and only if it is second countable.

Solution. (a) Suppose that X is second countable, and that $\mathcal{B} = \{B_n\}$ is a countable base for its topology. Let \mathcal{U} be

an open covering of X . For each x and each $U \in \mathcal{U}$ there is a $B_{x,U}$ in \mathcal{B} such that $x \in B_{x,U} \subset U$. The collection $\{B_{x,U} \mid x \in X, U \in \mathcal{U}\}$ is countable because it is a subcollection of base elements in \mathcal{B} . If we enumerate them like $B_{x_1, U_1}, B_{x_2, U_2}, \dots$, then $\{U_{x_1}, U_{x_2}, \dots\}$ is a subcovering of \mathcal{U} .

- (b) Let (X, d) be a metric space that is Lindelöf. For each $n = 1, 2, \dots$, the collection of open balls $\{B(x, 1/n) \mid x \in X\}$ is an open covering of X , and because X is Lindelöf, it admits a countable subcovering \mathcal{B}_n . Then $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ is a countable base for the topology of X . \square

¶ 9. Let ℓ^2 denote Hilbert space, that is, the space of square summable sequences of real numbers, $\ell^2 = \left\{ (x_n) \mid \sum_{n=1}^\infty x_n^2 < \infty \right\}$. It is a metric space with the distance function D defined by $D((x_n), (y_n)) = \left(\sum_{n=1}^\infty (x_n - y_n)^2 \right)^{1/2}$.

- (a) Prove that the closed balls $B(\mathbf{0}, \epsilon) = \left\{ (x_n) \in \ell^2 \mid \left(\sum_{n=1}^\infty x_n^2 \right)^{1/2} \leq \epsilon \right\}$ are not compact, and deduce that ℓ^2 is not locally compact.

Hilbert space ℓ^2 can be viewed as a subset of the countable product $\mathbf{R}^\mathbf{N}$. But the topology that it inherits from $\mathbf{R}^\mathbf{N}$ and the topology as metric space are distinct: ℓ^2 with the metric topology is not homeomorphic to ℓ^2 with the subspace topology from $\mathbf{R}^\mathbf{N}$. On the other hand, ℓ^2 and $\mathbf{R}^\mathbf{N}$ are homeomorphic (R. D. Anderson, *Bull. Amer. Math. Soc.* **72** (1966), 515–519.) The Hilbert cube \mathbf{I}^ω is the subset of ℓ^2 consisting of sequences (x_n) such that $0 \leq x_n \leq 1/n$ for all $n = 1, 2, 3, \dots$.

- (b) Prove that the mapping $f : \mathbf{I}^\omega \rightarrow [0, 1]^\mathbf{N}$ defined by $f(x_1, x_2, \dots) = (x_1, 2x_2, 3x_3, \dots)$ is bijective, continuous, and open. (Here \mathbf{I}^ω has the subspace topology from ℓ^2 , and $[0, 1]^\mathbf{N}$ has the product topology.)
- (c) Prove, using (b) or otherwise, that the Hilbert cube is compact.

¶ 10. Let X be Hausdorff and locally compact, and view X as a subspace of its one-point compactification X^* . Let $f : X \rightarrow \mathbf{R}$ be continuous. Prove that f admits a continuous extension to X^* (that is, there is $F : X^* \rightarrow \mathbf{R}$ continuous such that $F(x) = f(x)$ for all x in X) if and only if for each $\epsilon > 0$ there is a compact subset K_ϵ of X such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in X \setminus K_\epsilon$.

¶ 11. Chapter 2, Section 6: 1, 2, 3, 4, 5, 6, 8, 9, 10