¶ **1.** The space of real valued functions on $\mathbf{R}$ is the infinite product $\mathbf{R}^{\mathbf{R}}$. Prove the following:

(a) A sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ converges uniformly to a function f if $f_n \to f$ in the box topology.

(b) A sequence f_n converges pointwise to f if and only if $f_n \to f$ in the product topology.

¶ **2.** Any number x in the interval $[0, 1]$ has a ternary power series expansion of the form $x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}$, abbreviated $x = 0.a_1a_2a_3\cdots$. This expression is unique except for the fact that any number whose representation ends in a sequence of 2's has also has a representation ending in a sequence of 0's. For example, $1/3$ is represented by $0.100000\cdots$ and by $0.022222\cdots$.
Let C be the set of points x in the interval $[0, 1]$ admitting a ternary expansion which does not contain the digit 1, that is, $x \in C$ if $x = \sum a_n/3^n$, where each a_n is 0 or 2. The set C with the induced topology from the real line is called the *Cantor set*.

(a) Prove that C is homeomorphic to $X = \mathbf{2}^{\mathbf{N}}$, the product of countably many copies of the discrete two-point space $mathbf2 = \{0, 1\}$.

(b) Prove that the connected component of any $x \in C$ is $\{x\}$.

¶ **3.** Let $X = \mathbf{R}^{\mathbf{N}}$ be the set of sequences of real numbers.

(a) Let $f : \mathbf{R} \to \mathbf{R}^{\mathbf{N}}$ be the mapping given by $f(t) = (t, t, t, \cdots)$. Prove that f is continuous when $\mathbf{R}^{\mathbf{N}}$ has the product topology, but f is not continuous when $\mathbf{R}^{\mathbf{N}}$ has the box topology.

(b) Let A be the subset of X consisting of all sequences (x_n) such that $x_n \neq 0$ for only finitely many values of n. Find the closure of A in X in the product topology and in the box topology.

¶ **4.** Connectedness give a crude method for establishing that two spaces are not homeomorphic.

(a) $\mathbf{R}$ and R^n $(n > 1)$ are not homeomorphic.

(b) $\mathbf{R}$ and $[0, \infty)$ are not homeomorphic.

(c) $[0, 1]$ and the unit circle are not homeomorphic.

(d) The unit circle and the unit sphere in $\mathbf{R}^3$ are not homeomorphic.

¶ **5.** Two subsets A and B of a space X are said to be mutually separated if $A \cap B^- = A^- \cap B = \emptyset$. Prove that a subset S of X is connected if and only if there are no mutually separated subsets A and B of X such that $S = A \cup B$.

¶ **6.** A space X is path-connected if for any two points x_0 and x_1 in X there is a continuous mapping c from the interval $[0, 1]$ into X such that $c(0) = x_0$ and $c(1) = x_1$ (such c is called a path from x_0 to x_1. Prove the following:

(a) Continuous images of path-connected spaces are path connected.

(b) a non-empty product is path connected if and only if each factor is path connected.

(c) A path connected space is connected.

(d) $S = \{(x, \sin 1/x) \mid x > 0\} \cup \{(0, y) \mid -1 \leq y \leq 1\} \subset \mathbf{R}^2$ is connected but it is not path connected.

¶ **7.** Prove that the connected components of $\mathbf{Q}$ (endowed with the subspace topology from $\mathbf{R}$) are the points.

¶ **8.** Prove that if $E \subset X$ is connected and $E \subset A \subset E^-$, then A is also connected.

¶ **9.** True or False:

(a) $\mathbf{R}^{\mathbf{N}}$ with the box topology is connected.

(b) If $Y \subset X$ is path connected, then Y^- is path connected.

(c) the set of real numbers with the cofinite topology is connected.

¶ **10.** Let $X = \mathbf{R}^{\mathbf{N}}$ be the set of sequences of real numbers, endowed with the box topology. Prove that the component of $x = (x_n) \in X$ is the set of all sequences $y = (y_n)$ such that the set $\{n \mid x_n \neq y_n\}$ is finite.

¶ **11.** Let $\sim$ be the equivalence relation on $\mathbf{R}^2$ defined by $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_2 = y_2$ and $x_1 - y_1$ is an integer. The quotient space $X = \mathbf{R}^2/\sim$ is called a Mobius (infinite) band. Let $p : \mathbf{R}^2 \to X$ be the quotient map, and let $A \subset X$ be the image $p\{(x_1, 0) \mid x_1 \in \mathbf{R}\}$.

(a) Prove that A is a closed subset of X.

(b) Let $\sim_A$ be the equivalence relation on X given by $x \sim_A y$ if and only if either $x = y$, or $x, y \in A$ (or both). The quotient space $X/\sim_A$ is a familiar one, can you visualize it?