¶ **1.** The space of real valued functions on $\mathbf{R}$ is the infinite product $\mathbf{R}^{\mathbf{R}}$. Prove the following:

(a) A sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ converges uniformly to a function f if $f_n \to f$ in the box topology.

(b) A sequence f_n converges pointwise to f if and only if $f_n \to f$ in the product topology.

Solution. (a) Suppose that $f_n \to f$ in the box topology. If $\epsilon > 0$, then the open box $\prod_{x\in\mathbf{R}}(f(x)-\epsilon, f(x)+\epsilon)$ is a neighborhood of f in the box topology, so there is N such that $f_n \in \prod_{x\in\mathbf{R}}(f(x)-\epsilon, f(x)+\epsilon)$ for all $n > N$; that is, $|f(x) - f_n(x)| < \epsilon$ for all x in $\mathbf{R}$ and all $n > N$.

□

¶ **2.** Any number x in the interval $[0, 1]$ has a ternary power series expansion of the form $x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}$, abbreviated $x = 0.a_1a_2a_3\cdots$. This expression is unique except for the fact that any number whose representation ends in a sequence of 2's has also has a representation ending in a sequence of 0's. For example, $1/3$ is represented by $0.100000\cdots$ and by $0.022222\cdots$.
Let C be the set of points x in the interval $[0, 1]$ admitting a ternary expansion which does not contain the digit 1, that is, $x \in C$ if $x = \sum a_n/3^n$, where each a_n is 0 or 2. The set C with the induced topology from the real line is called the *Cantor set*.

(a) Prove that C is homeomorphic to $X = \mathbf{2}^{\mathbf{N}}$, the product of countably many copies of the discrete two-point space $\mathbf{2} = \{0, 1\}$.

(b) Prove that the connected component of any $x \in C$ is $\{x\}$.

¶ **3.** Let $X = \mathbf{R}^{\mathbf{N}}$ be the set of sequences of real numbers.

(a) Let $f : \mathbf{R} \to \mathbf{R}^{\mathbf{N}}$ be the mapping given by $f(t) = (t, t, t, \cdots)$. Prove that f is continuous when $\mathbf{R}^{\mathbf{N}}$ has the product topology, but f is not continuous when $\mathbf{R}^{\mathbf{N}}$ has the box topology.

(b) Let A be the subset of X consisting of all sequences (x_n) such that $x_n \neq 0$ for only finitely many values of n. Find the closure of A in X in the product topology and in the box topology.

Solution. The composition $\pi \circ f$ of f with any projection mapping $\pi : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ is given by $\pi \circ f(t) = t$ and so is a continuous mapping of $\mathbf{R}$ into $\mathbf{R}$. Therefore, f is continuous for the product topology.

The preimage of open box $\prod_{n\in\mathbf{N}}(-1/n, 1/n)$ is the set of all real numbers t such that $-1/n < t < 1/n$ for all n, that is, the set $\{0\}$, which is not open in $\mathbf{R}$. Therefore, f is not continuous for the box topology. □

¶ **4.** Connectedness provides a crude method for establishing that two spaces are not homeomorphic.

(a) $\mathbf{R}$ and $\mathbf{R}^n$ $(n > 1)$ are not homeomorphic.

(b) $\mathbf{R}$ and $[0, \infty)$ are not homeomorphic.

(c) $[0, 1]$ and the unit circle are not homeomorphic.

(d) The unit circle and the unit sphere in $\mathbf{R}^3$ are not homeomorphic.

Solution. Note that if $f : X \to Y$ is a homeomorphism, then for any $A \subset X$, the restriction mapping $f|_A : A \to f(A)$ is also a homeomorphism, where A and $f(A)$ are given the respective subspace topologies.

(a) The complement of a point in $\mathbf{R}$ is not connected, but the complement of a point in $\mathbf{R}^n$ is connected if $n > 1$ (any pair of points is contained in a connected subset.)

□

¶ **5.** Two subsets A and B of a space X are said to be mutually separated if $A \cap B^- = A^- \cap B = \emptyset$. Prove that a subset S of X is connected if and only if there are no mutually separated nonempty subsets A and B of X such that $S = A \cup B$.

Solution. If S is disconnected by A and B (that is, if A and B are two non-empty, disjoint, open subsets of S with $A \cup B = S$, then

$$A \cap \mathrm{Cl}_X B = (A \cap S) \cap \mathrm{Cl}_X B = A \cap (S \cap \mathrm{Cl}_X B) = A \cap \mathrm{Cl}_S B = A \cap B = \emptyset$$

Conversely, if A and B are mutually separated nonempty subsets of X and $S = A \cup B$, then

$$\mathrm{Cl}_S A = S \cap \mathrm{Cl}_X A = (A \cup B) \cap \mathrm{Cl}_X A = (A \cap \mathrm{Cl}_X A) \cup (B \cap \mathrm{Cl}_X A) = A$$

so that A is closed in S. A similar argument shows that B is closed in S, and thus A and B perform a disconnection of S. □

¶ **6.** A space X is path-connected if for any two points x_0 and x_1 in X there is a continuous mapping c from the interval $[0, 1]$ into X such that $c(0) = x_0$ and $c(1) = x_1$ (such c is called a path from x_0 to x_1. Prove the following:

(a) Continuous images of path-connected spaces are path connected.

(b) a non-empty product is path connected if and only if each factor is path connected.

(c) A path connected space is connected.

(d) $S = \{(x, \sin 1/x) \mid x > 0\} \cup \{(0, y) \mid -1 \le y \le 1\} \subset \mathbf{R}^2$ is connected but it is not path connected.

Solution. (a) Suppose that $f : X \to Y$ is continuous and X is path-connected. Note that $f : X \to f(X)$ is then also continuous (where $f(X) \subset Y$ has the subspace topology). Let y_0, y_1 be two points in $f(X)$. Then $y_0 = f(x_0)$ and $y_1 = f(x_1)$ for some x_0, x_1 in X. Id $c : [0, 1] \to X$ is continuous and satisfies $c(0) = x_0$ and $c(1) = x_1$, then the composition $f \circ c : [0, 1] \to f(X)$ is also continuous and satisfies $f \circ c(0) = y_0$ and $f \circ c(1) = y_1$.

(b) The space S is connected because it is the closure of the set $\{(x, \sin 1/x) \mid x > 0\}$, which is connected because it is homeomorphic to the interval $(0, \infty)$ via the projection mapping $(x, \sin 1/x) \mapsto x$.

Let $x_0 = (0, 0)$ and $x_1 = (1, 0)$ and suppose that $c : [0, 1] \to S$ is continuous with $c(0) = x_0$ and $c(1) = x_1$. Because c takes connected sets to connected sets, there are sequences a_n and b_n in $(0, 1)$ that converge to 0 and such that $\sin 1/a_n \to 1$ and $\sin 1/b_n \to -1$, as $n \to \infty$, and this makes it impossible for $\lim_{t\to 0} c(t)$ to exist. □

¶ **7.** Prove that the connected components of $\mathbf{Q}$ (endowed with the subspace topology from $\mathbf{R}$) are the points.

Proof. Let C be a subset of $\mathbf{Q}$. If there are p and q in C such that $p < q$, then there is an irrational number x such that $p < x < q$. It follows that C is disconnected because the sets $(-\infty, x) \cap C$ and $C \cap (x, \infty)$ are disjoint, nonempty, open in C, and have union equal to C. □

¶ **8.** Prove that if $E \subset X$ is connected and $E \subset A \subset E^-$, then A is also connected.

¶ **9.** True or False:

(a) $\mathbf{R}^{\mathbf{N}}$ with the box topology is connected.

(b) If $Y \subset X$ is path connected, then Y^- is path connected.

(c) the set of real numbers with the cofinite topology is connected.

Solution. (a) False

(b) False, cf. Problem 6.

(c) True. Any two nonempty open subsets have nonempty intersection. □

¶ **10.** Let $X = \mathbf{R}^{\mathbf{N}}$ be the set of sequences of real numbers, endowed with the box topology. Prove that the component of $x = (x_n) \in X$ is the set of all sequences $y = (y_n)$ such that the set $\{n \mid x_n \ne y_n\}$ is finite.

¶ **11.** Let $\sim$ be the equivalence relation on $\mathbf{R}^2$ defined by $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_2 = y_2$ and $x_1 - y_1$ is an integer. The quotient space $X = \mathbf{R}^2/\sim$ is called a Mobius (infinite) band. Let $p : \mathbf{R}^2 \to X$ be the quotient map, and let $A \subset X$ be the image $p\{(x_1, 0) \mid x_1 \in \mathbf{R}\}$.

(a) Prove that A is a closed subset of X.

(b) Let $\sim_A$ be the equivalence relation on X given by $x \sim_A y$ if and only if either $x = y$, or $x, y \in A$ (or both). The quotient space $X/\sim_A$ is a familiar one, can you visualize it?