¶ **1.** A subset D of a topological space X is dense in X if $D^- = X$.

(a) D is dense in X if and only if $D \cap U \neq \emptyset$ for every non-empty open subset $U \subset X$.

(b) If $\mathfrak{B}$ is a base for the topology of X, then D is dense in X if and only if D has non-empty intersection with every non-empty set in $\mathfrak{B}$.

(c) True or False: if $Y \subset X$ and D is dense in X, then $D \cap Y$ is dense in Y with the subspace topology.

(d) Prove that $D_1 \times D_2$ is dense in the product $X_1 \times X_2$ if and only if D_1 is dense in X_1 and D_2 is dense in X_2.

¶ **2.** A topological space X is first countable at x if there is a countable family $\mathfrak{N}$ of neighborhoods of x such that any neighborhood of x contains a neighborhood in the family $\mathfrak{N}$.

(a) Prove that if X is metrizable, then X is first countable at any of its points.

(b) Prove that if X is first countable at x and if $S \subset X$ contains x in its closure, then there is a sequence in S that converges to x.

¶ **3.** Prove the following:

(a) A space X has the discrete topology (i.e., any subset is open) if and only if all mappings $f : X \to Y$ are continuous, for all spaces Y.

(b) A space X has the trivial topology (i.e., the only open sets are $\emptyset$ and X) if and only all mappings $f : Y \to X$ are continuous, for all space Y.

¶ **4.** Let $f : X \to Y$ be a mapping of topological spaces. Prove the following:

(a) If $X = \bigcup_\alpha U_\alpha$, where each U_α is open in X, and the restrictions $f|_{U_\alpha}$ are continuous, then f is continuous.

(b) If $X = A \cup B$ where A and B are closed in X and the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous.

(c) What if $X = \bigcup_\alpha A_\alpha$, where each A_α is closed and each $f|_{A_\alpha}$ is continuous?

¶ **5.** Show that if E_k is closed in X_k for $k = 1, \cdots, n$, then $E_1 \times \cdots \times E_n$ is closed in $X_1 \times \cdots \times X_n$.

¶ **6.** Let A and B be subsets of topological spaces X and Y, respectively. Prove that in the product $X \times Y$ with the product topology the following identities hold:

(a) $(A \times B)^- = A^- \times B^-$

(b) $(A \times B)^\circ = A^\circ \times B^\circ$.

¶ **7.** A mapping from one topological space to a second topological space is *open* if it takes open sets onto open sets. A mapping is closed if it takes closed sets to closed sets (this was already defined in a previous homework on metric space).

(a) Prove that, for a product space $X = X_1 \times \cdots \times X_n$, the projection mappings $\pi_k : X \to X_k$ are open.

(b) Give an example showing that projections mappings are in general not closed. (For example, take $X = \mathbf{R} \times \mathbf{R}$ and $F = \{(x, y) \mid xy = 1\}$ in X.)

(c) Prove that $f : X \to Y$ is continuous and closed if and only if $f(A^-) = f(A)^-$ for every subset A of X.

¶ **8.** Let X be a topological space, let $p : X \to Y$ be onto, and let Y have the quotient topology induced by p.
Prove that $f : Y \to Z$ is continuous if and only if the composition $f \circ p$ is continuous.

¶ **9.** (a) Prove that if $f : X \to Y$ is continuous, onto, and either open or closed, then the topology on Y is the quotient topology induced by f.

(b) Find examples of quotient maps that: (i) are not open, (ii) are not closed, and (iii) are neither open nor closed.

¶ **10.** A subspace S of a topological space X is called a retract of X if there is a continuous map $r : X \to S$ such that $r(s) = s$ for all $s \in S$.

(a) Prove that every singleton in X (a subspace with exactly one point) is a retract of X.

(b) Prove that $[0, \infty)$ is a retract of $\mathbf{R}$.

(c) Suppose that X is an infinite set endowed with the cofinite topology. Prove that any open non-empty subset of X is a retract of X.

(d) Prove that $S \subset X$ is a retract of X if and only if for any space Y and any continuous mapping $f : S \to Y$ there is a continuous mapping $F : X \to Y$ such that $F(s) = f(s)$ for all s in S (such F is called an extension of f to X).