¶ **1.** A subset D of a topological space X is dense in X if $D^- = X$.

(a) D is dense in X if and only if $D \cap U \neq \emptyset$ for every non-empty open subset $U \subset X$.

(b) If $\mathfrak{B}$ is a base for the topology of X, then D is dense in X if and only if D has non-empty intersection with every non-empty set in $\mathfrak{B}$.

(c) True or False: if $Y \subset X$ and D is dense in X, then $D \cap Y$ is dense in Y with the subspace topology.

(d) Prove that $D_1 \times D_2$ is dense in the product $X_1 \times X_2$ if and only if D_1 is dense in X_1 and D_2 is dense in X_2.

Solution. (c) False: let $X = \mathbf{R}$ with the standard topology, let D be the subset of rational numbers and let Y be the complement of D. □

¶ **2.** A topological space X is first countable at x if there is a countable family $\mathfrak{N}$ of neighborhoods of x such that any neighborhood of x contains a neighborhood in the family $\mathfrak{N}$.

(a) Prove that if X is metrizable, then X is first countable at any of its points.

(b) Prove that if X is first countable at x and if $S \subset X$ contains x in its closure, then there is a sequence in S that converges to x.

Solution. (a) For any x in X, let $\mathfrak{N}_x = \{B(x, 1/n) \mid n = 1, 2, \cdots\}$.

(b) Enumerate $\mathfrak{N} = \{N_1, N_2, \cdots\}$. Let $N'_k = N_1 \cap N_2 \cdots \cap N_k$. Then $\{N'_k\}$ is a decreasing sequence ($N'_k \supset N'_{k+1}$) of neighborhoods of x. If x in the closure of S, then for any n there is x_k in $N'_k \cap S$. The sequence (x_k) converges to x because if U is a neighborhood of x then there is some $N_k \in \mathfrak{N}$ with $N_k \subset U$, and thus $N'_k \subset N_k \subset U$, which implies that $x_j \in U$ for all $j \geq k$ because $\{N'_k\}$ is a decreasing sequence.

□

¶ **3.** Prove the following:

(a) A space X has the discrete topology (i.e., any subset is open) if and only if all mappings $f : X \to Y$ are continuous, for all spaces Y.

(b) A space X has the trivial topology (i.e., the only open sets are $\emptyset$ and X) if and only all mappings $f : Y \to X$ are continuous, for all spaces Y.

Solution. (a) If S is a subset of X, let $\chi_S : X \to \mathbf{2}$ be the characteristic function of S. If χ_S is continuous, then $S = \chi_S^{-1}\{1\}$ is open.

(b) Let S be an open subset of X, let $\{0, 1\}$ have the trivial topology. If S is neither empty nor equal to X, then there are points $x \in S$ and $y \in X \setminus S$. The mapping $f : \{0, 1\} \to X$ given by $f(0) = x$ and $f(1) = y$ has $f^{-1}(S) = \{0\}$, so it cannot be continuous.

□

¶ **4.** Let $f : X \to Y$ be a mapping of topological spaces. Prove the following:

(a) If $X = \bigcup_\alpha U_\alpha$, where each U_α is open in X, and the restrictions $f|_{U_\alpha}$ are continuous, then f is continuous.

(b) If $X = A \cup B$ where A and B are closed in X and the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous.

(c) What if $X = \bigcup_\alpha A_\alpha$, where each A_α is closed and each $f|_{A_\alpha}$ is continuous?

¶ **5.** Show that if E_k is closed in X_k for $k = 1, \cdots, n$, then $E_1 \times \cdots \times E_n$ is closed in $X_1 \times \cdots \times X_n$.

¶ **6.** Let A and B be subsets of topological spaces X and Y, respectively. Prove that in the product $X \times Y$ with the product topology the following identities hold:

(a) $(A \times B)^- = A^- \times B^-$

(b) $(A \times B)^\circ = A^\circ \times B^\circ$.

Solution. (b) "$\supset$" $A^\circ \times B^\circ$ is open and contained in $A \times B$. "$\subset$" If $(x, y) \in (A \times B)^\circ$, then tere is an open set $W \subset A \times B$ such that $(x, y) \in U$. Then there is an open rectangle $U \times V \subset W$ containing (x, y). This implies that $x \in U \subset A$ and $y \in V \subset B$, that is, $x \in A^\circ$ and $y \in Y^\circ$. □

¶ **7.** A mapping from one topological space to a second topological space is *open* if it takes open sets onto open sets. A mapping is closed if it takes closed sets to closed sets (this was already defined in a previous homework on metric space).

(a) Prove that, for a product space $X = X_1 \times \cdots \times X_n$, the projection mappings $\pi_k : X \to X_k$ are open.

(b) Give an example showing that projections mappings are in general not closed. (For example, take $X = \mathbf{R} \times \mathbf{R}$ and $F = \{(x, y) \mid xy = 1\}$ in X.)

(c) Prove that $f : X \to Y$ is continuous and closed if and only if $f(A^-) = f(A)^-$ for every subset A of X.

Solution. (c) It was proven in class that f is continous if and only if $f(A^-) \subset f(A)^-$ for every $A \subset X$. If f is moreover closed, then $f(A^-)$ is a closed subset of Y that contains $f(A)$ (because $A \subset A^-$), hence $f(A)^- \subset f(A^-)$. Thus $f(A^-) = f(A)^-$. Conversely, if $f(A^-) = f(A)^-$ for every $A \subset X$, then f is continuous. It is also closed, because if $F \subset X$ is closed, then $f(F) = f(F^-) = f(F)^-$. □

¶ 8. Let X be a topological space, let $p : X \to Y$ be onto, and let Y have the quotient topology induced by p.
Prove that $f : Y \to Z$ is continuous if and only if the composition $f \circ p$ is continuous.

Solution. If f is continuous, then $f \circ p$ is the composite of continuous mapping and is therefore a continuous mapping. Conversely, suppose that $f \circ p$ is continuous. If $W \subset Z$ is open, then $(f \circ p)^{-1}W$ is open in X. Since $(f \circ p)^{-1}W = p^{-1}f^{-1}W$, that implies that $f^{-1}W$ is open in Y, by definition of the quotient topology. □

¶ 9. (a) Prove that if $f : X \to Y$ is continuous, onto, and either open or closed, then the topology on Y is the quotient topology induced by f.

(b) Find examples of quotient maps that: (i) are not open, (ii) are not closed, and (iii) are neither open nor closed.

¶ 10. A subspace S of a topological space X is called a retract of X if there is a continuous map $r : X \to S$ such that $r(s) = s$ for all $s \in S$.

(a) Prove that every singleton in X (a subspace with exactly one point) is a retract of X.

(b) Prove that $[0, \infty)$ is a retract of $\mathbf{R}$.

(c) Suppose that X is an infinite set endowed with the cofinite topology. Prove that any open non-empty subset of X is a retract of X.

(d) Prove that $S \subset X$ is a retract of X if and only if for any space Y and any continuous mapping $f : S \to Y$ there is a continuous mapping $F : X \to Y$ such that $F(s) = f(s)$ for all s in S (such F is called an extension of f to X).

Solution. (a) If $A = \{a\} \subset X$, let $r : X \to A$ be the mapping given by $r(x) = a$ for every x in X.

(b) Let $r : \mathbf{R} \to [0, \infty)$ be given by $r(x) = x$ if $x \geq 0$ and $r(x) = 0$ is $x \leq 0$.

(c) Let $x_0 \in U$ and let $r : X \to U$ be given by $r(x) = x$ if x is in U and $r(x) = x_0$ if x is not in U (there may not be any such x, that is, if $U = X$). If $U = X$ then r is the identity and thus continuous. If U is a proper subset of X, and if $V \subset U$ is open, then $r^{-1}V = V$ if $x_0 \neq inV$ and $r^{-1}V = V \cup \{x_0\}$, both open subset of X. □