¶ **1.** Prove that a uniformly continuous function takes Cauchy sequences to Cauchy sequences.

¶ **2.** Give an example of equivalent metrics d and d' on a set X such that there is a sequence in X that is Cauchy for d but it is not Cauchy for d'.

¶ **3.** (a) Prove that if (X, d) is separable, and d' is equivalent to d, then (X, d') is also separable.

(b) Find an example of two equivalent metrics d and d' on a set X such that (X, d) is complete but (X, d') is not complete.

¶ **4.** (a) Prove that X and Y are separable if and only if the product $X \times Y$ is separable (with any of d_1, d_2, or d_∞).

(b) Prove that X and Y are complete if and only if the product $X \times Y$ is complete (with any of d_1, d_2, or d_∞).

¶ **5.** Let x_0 be a fixed point in a metric space (X, d). Prove that the function $x \in X \mapsto d(x, x_0) \in \mathbf{R}$ is uniformly continuous.

¶ **6.** A mapping f from one metric space (X, d_X) to a second metric space (Y, d_Y) is Lipschitz if there is a constant $K \geq 0$ (a Lipschitz constant) such that $d_Y(f(x), f(z)) \leq K d_X(x, z)$, for all x and z in X.

(a) Prove that a Lipschitz mapping is uniformly continuous.

(b) Prove that if $f : \mathbf{R} \to \mathbf{R}$ has bounded derivative, then it is Lipschitz.

¶ **7.** If $f : X \to X$ is Lipschitz with Lipschitz constant $K < 1$, then f is called a contraction. The purpose of this problem is to prove the *Banach Fixed Point Theorem*: if X is complete and $f : X \to X$ is a contraction, then f has a unique fixed point (that is, there exists one and only one w in X such that $f(w) = w$).
Assume thus that X is complete and that $f : X \to X$ is a contraction.

(a) Prove that if x and y are both fixed points of f, then $x = y$.

Let x_0 be any point in x and construct a sequence x_0, $x_1 = f(x_0)$, $x_2 = f(x_1)$, $\ldots$, $x_n = f(x_{n-1}$ and so on.

(b) Prove that $d(x_{n+1}, x_n) \leq K d(x_n, x_{n-1})$.

(c) Prove that $d(x_p, x_q) \leq K^p \dfrac{K^{q-p} - 1}{K - 1} d(x_0, x_1)$, for any $1 \leq p < q$. (Use (b) and the iterated form of the triangle inequality as in Problem 2 of Homework 1.)

(c) Use (c) to prove that $\{x_n\}$ is a Cauchy sequence in X.

(e) Prove that the limit $\lim_n x_n$ exists and is the unique fixed point of f.

¶ **8.** Let $X = [1, \infty) \subset \mathbf{R}$ with the induced metric, and let $f : X \to X$ be $f(x) = x + 1/x$. Prove that $d(f(x), f(y)) < d(x, y)$ for any $x \neq y$, but f has no fixed points.

¶ **9.** Let X be the set of bounded sequences of real numbers with the metric $d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n|$. Let $f, g : X \to X$ be defined by $f(\{x_1, x_2, x_3, \cdots\}) = \{x_2, x_3, \cdots\}$ and $g(\{x_1, x_2, x_3, \cdots\}) = \{0, x_1, x_2, x_3, \cdots\}$.

(a) Prove that f and g are uniformly continuous.

(b) Find all the fixed points of f and of g.

¶ **10.** Let (X, d) be a metric space and let $\mathcal{F}$ be the family of all non-empty closed and bounded subsets of X. For A, B in $\mathcal{F}$, define

$$d_H(A, B) = \max\{\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)\}.$$

(Refer to Problem 5(a) of Homework 3 for the definition of $d(a, B)$), Then d_H is a metric on $\mathcal{F}$ called the Hausdorff metric.

(a) Prove that for any x, z in X, $d(x, z) = d_H(\{x\}, \{z\})$.

(b) Let $X = \mathbf{R}^2$. Let $A_0 = \{(0, 0)\}$, $A_1 = \{(1, 0)\}$, A_2 the unit circle with center at the origin, and A_3 the equilateral triangle inscribe in the unit circle A_2 with one vertex at $(1, 0)$. Evaluate the Hausdorff distances $d_H(A_i, A_j)$.