¶ **1.** Prove that a set is open if and only if it is a neighborhood of each of its points.

¶ **2.** Prove that, in any metric space,

(a) every closed set is the intersection of a countable number of open sets.

(b) every open set is the union of a countable number of closed sets.

¶ **3.** Prove that f is continuous if and only if it sends convergent sequences to convergent sequences.

¶ **4.** Let f, g be a real valued continuous mappings on the metric space X.

(a) Prove that if $f(x_0) > 0$ then there is $r > 0$ such that $f(x) > 0$ for all x in the ball $B(x_0, r)$.

(b) Prove that the mappings $f + g$, $f - g$, $f \cdot g$, $|f|$, $\max\{f, g\}$, $\min\{f.g\}$, and f/g if $g(x) \neq 0$ for all x, are continuous on X.

¶ **5.** Let (X, d) be a metric space.

(a) Let x_0 be a fixed point in X. Prove that the function $f : X \to \mathbf{R}$ defined by $f(x) = d(x, x_0)$ is continuous.

(b) Let $A \subset X$. Prove that the function $g(x) = d(x, A)$ is continuous, where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$

¶ **6.** If $f : X \to Y$ is continuous at x, and $g : Y \to Z$ is continuous at $f(x)$, then $g \circ f$ is continuous at x.

¶ **7.** Let X, Z be metric space and let d_2, d_1, and d_∞ be the metrics on $X \times Z$ given by

$$\begin{aligned} d_2((x,z),(x',z')) &= \sqrt{d_X(x,x')^2 + d_Z(z,z')^2} \\ d_1((x,z),(x',z')) &= d_X(x,x') + d_Z(z,z') \\ d_\infty((x,z),(x',z')) &= \max\{d_X(x,x'), d_Z(z,z')\} \end{aligned}$$

(a) Prove that d_2, d_1 and d_∞ are equivalent metrics on $X \times Z$.

(b) Prove that the mappings $p_X : X \times Z \to X$ and $p_Z : X \times Z \to Z$ given by $p_X(x, z) = x$ and $p_Z(x, z) = z$, respectively, are continuous, where $X \times Z$ is endowed with any of the metrics d_2, d_1, or d_∞.

(c) Let (x_0, z_0) be a fixed point in $X \times Z$. Prove that $i : Z \to X \times Z$ and $j : X \to X \times Z$ given by $i(z) = (x_0, z)$ and $j(x) = (x, z_0)$ are continuous.

¶ **8.** A mapping from one metric space to a second metric space is called open if it sends open subsets into open sets.

(a) Give an example of a continuous mapping that is not open.

(b) Give an example of an open mapping that is not continuous.

(c) Prove that if $f : X \to Y$ is a bijection with inverse $f^{-1} : Y \to X$, then f is open if and only if f^{-1} is continuous.

(d) Let X be a set with two metrics d_1 and d_2. Prove that d_1 and d_2 are equivalent if and only if the identity mapping $\mathrm{id} : (X, d_1) \to (X, d_2)$, given by $\mathrm{id}(x) = x$, is open and continuous.

¶ **9.** Let $f : X \to Y$ be continuous.

(a) Prove that the set $B = \{x \in X \mid f(x) = g(x)\}$ is a closed subset of x.

(b) Prove that if $f(x) = g(x)$ for all x in a dense subset of X, then $f = g$ in X.

¶ **10.** For a mapping $f : X \to Z$, let $G(f) = \{(x, z) \in X \times Z \mid f(x) = z\}$ be the graph of f.

(a) Prove that if f is continuous, then $G(f)$ is a closed in $X \times Z$, where $X \times Z$ is endowed with any of the metrics d_2, d_1 or d_∞ from Problem 7.

(b) Is the converse true?