

¶ 1. Prove that a set is open if and only if it is a neighborhood of each of its points.

¶ 2. Prove that, in any metric space,

(a) every closed set is the intersection of a countable number of open sets.

(b) every open set is the union of a countable number of closed sets.

Solution. (a) Let Y be a subset of the metric space. For $n = 1, 2, 3, \dots$, the set $U_n = \bigcup_{y \in Y} B(y, 1/n)$ is open and contains Y , and the intersection $\bigcap_{n=1}^{\infty} U_n = \bar{Y}$. □

¶ 3. Prove that f is continuous if and only if it sends convergent sequences to convergent sequences.

Hint. We must show that if x_n is a convergent sequence in X and $x_n \rightarrow x$, then $f(x_n)$ converges to $f(x)$. Indeed, x_1, x, x_2, x, \dots also converges to x , and if the sequence $f(x_1), f(x), f(x_2), f(x), \dots$ must converge, it must do so to $f(x)$. □

¶ 4. Let f, g be a real valued continuous mappings on the metric space X .

(a) Prove that if $f(x_0) > 0$ then there is $r > 0$ such that $f(x) > 0$ for all x in the ball $B(x_0, r)$.

(b) Prove that the mappings $f + g, f - g, f \cdot g, |f|, \max\{f, g\}, \min\{f, g\}$, and f/g if $g(x) \neq 0$ for all x , are continuous on X .

¶ 5. Let (X, d) be a metric space.

(a) Let x_0 be a fixed point in X . Prove that the function $f : X \rightarrow \mathbf{R}$ defined by $f(x) = d(x, x_0)$ is continuous.

(b) Let $A \subset X$. Prove that the function $g(x) = d(x, A)$ is continuous, where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$

¶ 6. If $f : X \rightarrow Y$ is continuous at x , and $g : Y \rightarrow Z$ is continuous at $f(x)$, then $g \circ f$ is continuous at x .

¶ 7. Let X, Z be metric space and let d_2, d_1 , and d_∞ be the metrics on $X \times Z$ given by

$$\begin{aligned} d_2((x, z), (x', z')) &= \sqrt{d_X(x, x')^2 + d_Z(z, z')^2} \\ d_1((x, z), (x', z')) &= d_X(x, x') + d_Z(z, z') \\ d_\infty((x, z), (x', z')) &= \max\{d_X(x, x'), d_Z(z, z')\} \end{aligned}$$

(a) Prove that d_2, d_1 and d_∞ are equivalent metrics on $X \times Z$.

(b) Prove that the mappings $p_X : X \times Z \rightarrow X$ and $p_Z : X \times Z \rightarrow Z$ given by $p_X(x, z) = x$ and $p_Z(x, z) = z$, respectively, are continuous, where $X \times Z$ is endowed with any of the metrics d_2, d_1 , or d_∞ .

(c) Let (x_0, z_0) be a fixed point in $X \times Z$. Prove that $i : Z \rightarrow X \times Z$ and $j : X \rightarrow X \times Z$ given by $i(z) = (x_0, z)$ and $j(x) = (x, z_0)$ are continuous.

¶ 8. A mapping from one metric space to a second metric space is called open if it sends open subsets into open sets.

(a) Give an example of a continuous mapping that is not open.

(b) Give an example of an open mapping that is not continuous.

(c) Prove that if $f : X \rightarrow Y$ is a bijection with inverse $f^{-1} : Y \rightarrow X$, then f is open if and only if f^{-1} is continuous.

(d) Let X be a set with two metrics d_1 and d_2 . Prove that d_1 and d_2 are equivalent if and only if the identity mapping $\text{id} : (X, d_1) \rightarrow (X, d_2)$, given by $\text{id}(x) = x$, is open and continuous.

¶ 9. Let $f : X \rightarrow Y$ be continuous.

- (a) Prove that the set $B = \{x \in X \mid f(x) = g(x)\}$ is a closed subset of X .
- (b) Prove that if $f(x) = g(x)$ for all x in a dense subset of X , then $f = g$ in X .

Solution. (a) Let $x \in \overline{B}$. Then there is a sequence x_n on B that converges to x . Because f and g are continuous, $\lim f(x_n) = f(x)$ and $\lim g(x_n) = g(x)$, and because x_n is in B , $f(x_n) = g(x_n)$, hence $f(x) = g(x)$. That is, x is also in B .

- (b) B is closed (by (a)) and dense, thus $B = X$.

□

¶ 10. For a mapping $f : X \rightarrow Z$, let $G(f) = \{(x, z) \in X \times Z \mid f(x) = z\}$ be the graph of f .

- (a) Prove that if f is continuous, then $G(f)$ is a closed in $X \times Z$, where $X \times Z$ is endowed with any of the metrics d_2, d_1 or d_∞ from Problem 7.
- (b) Is the converse true?

Proof. (a) Since the metrics d_1, d_2 and d_∞ on $X \times Z$ are equivalent, they have the same open sets, so the same closed sets. We can use any of them to solve this problem. With d_∞ it is very easy to verify that a sequence $(x_n, z_n) \rightarrow (x, z)$ in $X \times Z$ if and only if $x_n \rightarrow x$ in X and $z_n \rightarrow z$ in Z .

Let (x, z) be a point in the closure of $G(f)$. Then there is a sequence (x_n, z_n) of points in $G(f)$ such that $(x_n, z_n) \rightarrow (x, z)$. This implies that $f(x_n) = z_n$, and also, by what was said in the previous paragraph, that $x_n \rightarrow x$ and $z_n \rightarrow z$. Since f is continuous, $\lim_n f(x_n) = f(x)$, and thus $f(x) = z$. That is, $(x, z) \in G(f)$.

- (b) Take $X = Z = \mathbf{R}$ and the map $f : X \rightarrow Z$ given by $f(x) = 1/x$ if $x \neq 0$, and $f(0) = 1$

□