¶ **1.** Prove or provide a counterexample:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(c) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

(d) $\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$

(e) $\overline{\bigcap_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$

¶ **2.** Let Y be a subset of a metric space X. Prove that for any subset S of Y, the closure of S in Y coincides with $\overline{S} \cap Y$, where $\overline{S}$ is the closure of S in X.

¶ **3.** (a) If $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ both converge to x, then the sequence $x_1, y_1, x_2, y_2, x_3, y_3, \cdots$ also converges to x.

(b) If $x_1, x_2, x_3, \cdots$ converges to x and $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection, then $x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}$ also converges to x.

(c) If $\{x_n\}$ is a sequence that does not converge to y, then there is an open ball $B(y, r)$ and a subsequence of $\{x_n\}$ outside $B(y, r)$.

(d) If $\{x_n\}$ is a sequence such that every of its subsequences has a subsequence that converges to x, then $\{x_n\}$ converges to x.

¶ **4.** Let X be a metric space.

(a) A point x in X is a limit point of a subset S of X if every ball $B(x, r)$ contains infinitely many points of S. Prove that x is a limit point of S if and only if there is a sequence $x_1, x_2, \cdots$ of points in S such that $\lim_{n\to\infty} x_n = x$ and $x_n \neq x$ for all n.

(b) Prove that the set of limit points of S is a closed set

(c) A point x is an isolated point of S if there is $r > 0$ such that $B(x, r) \cap S = \{x\}$. Prove that the closure of any subset S of X is the disjoint union of the set of limit points of S and the set of isolated points of S.

¶ **5.** Two metrics on a set X are equivalent if they determine the same open sets.

(a) Prove that two metrics d and d' on X are equivalent if and only if convergent sequences in d are the same as convergent sequences in (X, d').

(b) Prove that the metrics d (standard), d_1 and d_∞ on $\mathbf{R}^2$ are all equivalent.

¶ **6.** A metric space (X, d) is called discrete the metric d is equivalent to the discrete metric.

(a) Prove that a metric space is discrete if and only if it has no limit points.

(b) Prove that a metric space is discrete every convergent sequence is ultimately constant.

(c) Prove that a metric space in which the closure of any open set is open is discrete.

¶ **7.** A set of the form $\{z \in X \mid d(x, z) \leq r\}$ in a metric space X is called a closed ball (with center x and radius r).

(a) Prove that a closed ball is a closed set.

(b) Is the closed ball $\{z \in X \mid d(x, z) \leq r\}$ the closure of the open ball $B(x, r)$?

(c) Prove that for any x and any $r, s \geq 0$, the set $\{z \in X \mid s \leq d(x, z) \leq r\}$ is closed.

¶ **8.** Let Y be a subset of a metric space X.

(a) Prove that the interior of Y is the largest open subset of X that is contained in Y. (Hint: prove that the interior of Y is the union of all open subsets of X that are contained in Y.)

(b) Prove that the closure of Y is the smallest closed subset of X that contains Y.

¶ **9.** Let Y be a dense subset of a metric space. Suppose that every Cauchy sequence in Y converges to a point in X. Prove that X is complete.

¶ **10.** Prove that if A and B are complete metric spaces then $A \cup B$ and $A \cap B$ are also complete.

¶ **11.** Let X be the set of all bounded sequences of real numbers, with the distance given by

$$(\{x_n\}, \{z_n\}) = \sup_n |x_n - y_n|.$$

Prove that (X, d) is complete.

¶ **12.** (a) Prove that the closure of a nowhere dense set is nowhere dense.

(b) Prove that a union of finitely many nowhere dense sets is nowhere dense.

(c) Prove that a nowhere dense contains no isolated points.