¶ **1.** Prove or provide a counterexample:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(c) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

(d) $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

(e) $\overline{\bigcap_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$

Solution. (a) True.

(b) True. Let x be in $\overline{A} \cup \overline{B}$. Then x is in $\overline{A}$ or in $\overline{B}$, or in both. If x is in $\overline{A}$, then $B(x, r) \cap A \neq \emptyset$ for all $r > 0$, and so $B(x, r) \cap (A \cup B) \neq \emptyset$ for all $r > 0$; that is, x is in $\overline{A \cup B}$. For the reverse containment, note that $\overline{A} \cup \overline{B}$ is a closed set that contains $A \cup B$, there fore, it must contain the closure of $A \cup B$.

(c) False. Take $A = (0, 1)$, $B = (1, 2)$ in $\mathbf{R}$.

(d) False. Take $A_n = [1/n, 1 - 1/n]$, $n = 2, 3, \cdots$, in $\mathbf{R}$.

(e) False.

□

¶ **2.** Let Y be a subset of a metric space X. Prove that for any subset S of Y, the closure of S in Y coincides with $\overline{S} \cap Y$, where $\overline{S}$ is the closure of S in X.

¶ **3.** (a) If $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ both converge to x, then the sequence $x_1, y_1, x_2, y_2, x_3, y_3, \cdots$ also converges to x.

(b) If $x_1, x_2, x_3, \cdots$ converges to x and $\sigma : \mathbf{N} \to \mathbf{N}$ is a bijection, then $x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \cdots$ also converges to x.

(c) If $\{x_n\}$ is a sequence that does not converge to y, then there is an open ball $B(y, r)$ and a subsequence of $\{x_n\}$ outside $B(y, r)$.

(d) If $\{x_n\}$ is a sequence such that every of its subsequences has a subsequence that converges to x, then $\{x_n\}$ converges to x.

Solution. (b) Let $\epsilon > 0$. Because $x_n \to x$, there is N such that $d(x_n, x) < \epsilon$ if $n > N$. Because σ is a bijection, there is N' such that $\sigma(n) > N$ if $n > N'$. Hence, $d(x_{\sigma(n)}, x) < \epsilon$ if $n > N'$.

(c) Suppose that $\{x_n\}$ does not converge to x. Then there is an $\epsilon > 0$ with the property that for any N, there is $n \geq N$ such that $d(x_n, x) \geq \epsilon$. Thus, for $N = 1, 2, 3, \cdots$, there are $n_1 < n_2 < n_2 < \cdots$ such that $n_N > N$ and $d(x_{n_N}, x) \geq \epsilon$,

(d) If x_n does not converge to x, then, by (c), there is a subsequence $\{x_{n_k}\}$ such that $d(x_{n_k}, x) \geq r > 0$; thus this subsequence $\{x_{n_k}\}$ cannot have a subsequence that converges to x.

□

¶ **4.** Let X be a metric space.

(a) A point x in X is a limit point of a subset S of X if every ball $B(x, r)$ contains infinitely many points of S. Prove that x is a limit point of S if and only if there is a sequence $x_1, x_2, \cdots$ of points in S such that $\lim_{n \to \infty} x_n = x$ and $x_n \neq x$ for all n.

(b) Prove that the set of limit points of S is a closed set

(c) A point x is an isolated point of S if there is $r > 0$ such that $B(x, r) \cap S = \{x\}$. Prove that the closure of any subset S of X is the disjoint union of the set of limit points of S and the set of isolated points of S.

¶ **5.** Two metrics on a set X are equivalent if they determine the same open sets.

(a) Prove that two metrics d and d' on X are equivalent if and only if convergent sequences in d are the same as convergent sequences in (X, d').

(b) Prove that the metrics d (standard), d_1 and d_∞ on $\mathbf{R}^2$ are all equivalent.

¶ **6.** A metric space (X, d) is called discrete if the metric d is equivalent to the discrete metric.

(a) Prove that a metric space is discrete if and only if it has no limit points.

(b) Prove that a metric space is discrete if and only if every convergent sequence is ultimately constant.

(c) Prove that a metric space in which the closure of any open set is open is discrete.

Solution. (a) If X is discrete, then any subset of X is open. Therefore, for every x in X there is $r_x > 0$ such that $B(x, r_x) = \{x\}$ and so x cannot be a limit point of X.

If X has no limit point, then for any x there is $r > 0$ such that $B(x)$ contains only finitely many points, say $x_1, x_2, \cdots, x_n$. If $r_x = \min\{d(x, x_j) \mid x \neq x_j, j = 1, \cdots, n\}$, then $B(x, r_x) = \{x\}$. Therefore, each singleton $\{x\}$ is an open set, and so any open subset of X is an open set.

(b) Use Problem 5

(c) Suppose that x is a limit point, that is, for any $r > 0$ the ball $B(x, r)$ contains at least two distinct points. Find a sequence $x_1, x_2, \cdots$ that converges to x, such that $d(x, x_1) > d(x, x_2) > \cdots$. This property guarantees that all distances $d(x_n, x_m) \neq 0$ if $n \neq m$, and that $\inf_{n\neq m} d(x_n, x_m) = d_n > 0$. Let $B_n = B(x_n, d_n/3)$. The $U \bigcup_{n=1}^{\infty} B_{2n}$ is an open set such that $x \in \overline{U}$. Therefore, by the hypothesis, $\overline{U}$ is an open set containing x. But by construction, $\overline{U}$ does not contain any of the terms x_{2n+1}, contradicting that the sequence x_{2n+1} converges to x.

□

¶ **7.** A set of the form $\{z \in X \mid d(x, z) \leq r\}$ in a metric space X is called a closed ball (with center x and radius r).

(a) Prove that a closed ball is a closed set.

(b) Is the closed ball $\{z \in X \mid d(x, z) \leq r\}$ the closure of the open ball $B(x, r)$?

(c) Prove that for any x and any $r, s \geq 0$, the set $\{z \in X \mid s \leq d(x, z) \leq r\}$ is closed.

Solution. (a) If z is not in the closed ball with center x and radius r, then $d(x, z) > r$. Then $s = d(x, z) - r > 0$ and the ball $B(z, s)$ is outside that closed ball.

□

¶ **8.** Let Y be a subset of a metric space X.

(a) Prove that the interior of Y is the largest open subset of X that is contained in Y.

(b) Prove that the closure of Y is the smallest closed subset of X that contains Y.

Solution. (a) The interior, Y°, of Y is open and is contained in Y. Let $U \subset Y$ be an open set. If x is in U, then there is a ball $B(x, r) \subset U$, hence $B(x, r) \subset Y$. This means, by definition, that $x \in Y^\circ$.

□

¶ **9.** Let Y be a dense subset of a metric space. Suppose that every Cauchy sequence in Y converges to a point in X. Prove that X is complete.

¶ **10.** Prove that if A and B are complete metric spaces then $A \cup B$ and $A \cap B$ are also complete.

¶ **11.** Let X be the set of all bounded sequences of real numbers, with the distance given by

$$d(\{x_k\}, \{z_k\}) = \sup_n |x_k - y_k|.$$

Prove that (X, d) is complete.

Proof. Let $\mathbf{x}_n = \{x_{n,1}, x_{n,2}, \cdots\}$ be a sequence of points in X (a sequence of sequences) that is Cauchy for the metric d. Then, for every $\epsilon > 0$, there is $N > 0$ such that $d(\mathbf{x}_n, \mathbf{x}_m) < \epsilon$, if $n, m > N$. By definition, $d(\mathbf{x}_n, \mathbf{x}_m) = \sup_k |x_{n,k} - x_{m,k}|$, so it follows that, for each $k = 1, 2, \cdots$, the sequence of real numbers $\{x_{1,k}, x_{2,k}, x_{3,k}, \cdots\}$ is Cauchy. Therefore, since $\mathbf{R}$ is complete, the limit $\lim_{n\to\infty} x_{n,k}$ exists, for every k. Call $z_k = \lim_{n\to\infty} x_{n,k}$.
Two things must be shown: (a) $\mathbf{z} = \{z_k\}$ is a bounded sequence (so that it is a point in X), and (b) $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{z}$ in X.

(a) For $\epsilon = 1$, the Cauchy property of $\{\mathbf{x}_n\}$ implies that $d(\mathbf{x}_n, \mathbf{x}_m) < 1$ for all $n, m > N$. Hence, if $r = 2\max\{1, d(\mathbf{x}_1, \mathbf{x}_{N+1}), \cdots, d(\mathbf{x}_N, \mathbf{x}_{N+1})\}$, then $d(\mathbf{x}_p, \mathbf{x}_q) \le r$ for all p, q, and so, by the triangle inequality, $\sup_{n,k} |x_{n,k}| \le r$. Therefore, the sequence $\mathbf{z} = \{z_k\}$ satisfies $|z_k| \le r$, so it is bounded.

(b) Given $\epsilon > 0$, let $N > 0$ be such that $\sup_k |x_{n,k} - x_{m,k}| < \epsilon/2$ for $n, m > N$. Then,

$$|x_{n,k} - z_k| \le |x_{n,k} - x_{m,k}| + |x_{m,k} - z_k| < \epsilon/2 + |x_{m,k} - z_k|,$$

for all k and for all $n, m > N$. Take the limit as $m \to \infty$ on both side of the inequality to obtain that, for all k,

$$\lim_{m\to\infty} |x_{n,k} - z_k| = |x_{n,k} - z_k| \le \epsilon/2 + \lim_{m\to\infty} |x_{m,k} - z_k| = \epsilon/2,$$

because the left side $|x_{n,k} - z_k|$ does not depend on m and the sequence $x_{m,k} \to z_k$ as $m \to \infty$. Therefore, if $n > N$, then

$$d(\mathbf{x}_n, z) = \sup_k |x_{n,k} - z_k| \le \epsilon/2 < \epsilon.$$

□

¶ **12.** (a) Prove that the closure of a nowhere dense set is nowhere dense.

(b) Prove that a union of finitely many nowhere dense sets is nowhere dense.

(c) Prove that a nowhere dense set in a metric space contains no isolated points of the metric space.

Proof. (a) Let A be nowhere dense, and let $B = \overline{A}$. Then $\overline{B} = \overline{\overline{A}} = \overline{A}$, so the interior of $\overline{B}$ is the interior of $\overline{A}$, which is empty.

(b) Suppose that A_1 and A_2 are nowhere dense, and let $A = A_1 \cup A_2$. By Problem 1, $\overline{A} = \overline{A_1} \cup \overline{A_2}$. It must be shown that $\mathrm{Int}(A) = \emptyset$. We have that $\mathrm{Int}\,\overline{A_1} = \mathrm{Int}\,\overline{A_2} = \emptyset$, but note that in general, the union of the interiors of two sets is a proper the interior of the union of those sets.

Let B be an open set contained in $\overline{A_1} \cup \overline{A_2}$. If x is in B, then any ball around x contained in B must contain points in both $\overline{A_1}$ and in $\overline{A_2}$ because if there was a ball around x that contained no points of one of them, say $\overline{A_1}$, the that ball would be contained in $\overline{A_2}$, contradicting that $\mathrm{Int}\,\overline{A_2} = \emptyset$. This shows that any open set B in $\overline{A_1} \cup \overline{A_2}$ is contained in $\overline{A_1} \cap \overline{A_2}$, and hence that $B = \emptyset$.

This proof extends without difficulty to the case of finitely many nowhere dense sets. Is it true that a union of countably many nowhere dense sets is nowhere dense?

(c) Let A be nowhere dense and let x be is an isolated point of X, Then there is $r > 0$ such that $B(x, r) = \{x\}$. If $x \in A$, then $\{x\} = B(x, r) \subset A \subset \overline{A}$, so that $x \in \mathrm{Int} \equiv \emptyset$, a contradiction.

□