¶ **1.** Prove that for any three points x, z, w in a metric space we have

$$|d(x, w) - d(z, w)| \leq d(x, z).$$

Solution. Apply the triangle inequality twice to obtain $-d(x, z) \leq d(x, w) - d(z, w) \leq d(x, z)$. □

¶ **2.** Prove that in any metric space $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$.

Hint. Iterate the triangle inequality. □

¶ **3.** Let X be a set and $d : X \times X \to \mathbf{R}$ be a function such that $d(x, x) = 0$, and for $x \neq z$, $d(x, z) = d(z, x)$ is a number between 1 and 2 (which may depend on x and z). Prove that d is a metric on X.

Hint. The triangle inequality follows from the fact that if $1 \leq r, s \leq 2$, then $2 \leq r + s$. □

¶ **4.** Let X be a set and $d : X \times X \to \mathbf{R}$ be a function that satisfies $d(x, x) = 0$, $d(x, z) \neq 0$ for $x \neq z$, and $d(x, z) \leq d(z, w) + d(w, x)$. Prove that d is a metric on X.

Solution. Take $x = w$ in $d(x, z) \leq d(z, w) + d(w, x)$ and obtain $d(x, z) \leq d(z, x)$; take $x = w$ in $d(z, x) \leq d(x, w) + d(w, z)$ and obtain $d(z, x) \leq d(x, z)$. Therefore $d(x, z) = d(z, x)$. Finally, let $x = w$ in $d(x, w) \leq d(w, z) + d(z, x)$ to obtain $0 \leq d(x, z) + d(z, x) = 2d(x, z)$, or $d(x, z) \geq 0$. □

¶ **5.** Let d be a metric on X. Prove that $\dfrac{d}{1 + d}$ and $\min\{1, d\}$ are also metrics on X.

Hint. Proving the triangle inequality for $d/1 + d$ is equivalent to proving that $0 < x \leq z + w$ implies

$$\frac{x}{1 + x} \leq \frac{z}{1 + z} + \frac{w}{1 + w},$$

and this inequality is equivalent to

$$\frac{1}{1 + z} + \frac{1}{1 + w} \leq 1 + \frac{1}{1 + x}.$$

But if $x \leq z + w$, then

$$\begin{aligned} \frac{1}{1 + z} + \frac{1}{1 + w} &= \frac{2 + z + w}{1 + z + w + zw} \\ &\leq \frac{2 + z + w}{1 + z + w} \\ &= 1 + \frac{1}{1 + z + w} \\ &\leq 1 + \frac{1}{1 + x} \end{aligned}$$

□

¶ **6.** Let X be the set of all sequences $x = \{x_n\}_{n=0}^{\infty}$ of real numbers, and let $d : X \times X \to \mathbf{R}$ be the function

$$d(x, z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - z_n|}{1 + |x_n - z_n|}.$$

Prove that d is a metric on X.

Hint. Use Problem 5. □

¶ **7.** Let d_1 and d_2 be metrics on X. Which of the following are metrics on X: (a) $d_1 + d_2$; (b) $\max\{d_1, d_2\}$; (c) $\min\{d_1, d_2\}$?

¶ **8.** Let (X, d_Y) and (Z, d_Z) be metric spaces. Prove that the Cartesian product $X \times Z$ is a metric space with

$$d((x_1, z_1), (x_2, z_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Z(z_1, z_2)^2}.$$

Solution. Only the triangle inequality requires some work. Let (x_1, z_1), (x_2, z_2) and $(x, 3, z_3)$ be three points in $X \times Z$ and let $A = d(x_1, x_3)$, $B = d(z_1, z_3)$, $a = d(x_1, x_2)$, $\alpha = d(x_2, x_3)$, $b = d(z_1, z_2)$ and $\beta = d(z_2, z_3)$. Then

$$A \leq a + \alpha \quad \text{and} \quad B \leq b + \beta$$

by the triangle inequality for d_X and D_Z. Proving the triangle inequality for d is equivalent to proving that

$$\sqrt{A^2 + B^2} \leq \sqrt{a^2 + b^2} + \sqrt{\alpha^2 + \beta^2}$$

Squaring both sides and using that $A \leq a + \alpha$ and $B \leq b + \beta$, this will follow from establishing the inequality

$$(a + \alpha)^2 + (b + \beta)^2 \leq a^2 + b^2 + \alpha^2 + beta^2 + 2\sqrt{(a^2 + b^2)(\alpha^2 + \beta^2)}$$

which is equivalent to

$$a\alpha + b\beta \leq \sqrt{(a^2 + b^2)(\alpha^2 + \beta^2)},$$

which is obviously true (square both sides). □

¶ **9.** Let X be the set of finite words in an alphabet. For two words x, z in X, let $D(x, z)$ be the minimum number of edit operations needed to transform the word x into the word z, where an edit operation is an insertion, deletion, or substitution of a single character of the alphabet. Prove that D is a metric on X.

¶ **10.** (a) Prove that in any metric space the complement of a point is an open set.

(b) Prove that any set in a metric space is an intersection of open sets.

Solution. (a) Let $U = X \setminus \{x\}$ be the complement of the point x in X. If $z \in U$, then $d(x, z) > 0$ and the ball $B(z, d(x, z)) \subset U$.

(b) Let $S \subset X$. Then $S = \bigcap_{x \in X \setminus S} X \setminus \{x\}$. Indeed, if x is in $X \setminus S$, then $S \subset X \setminus \{x\}$, and thus, $S \subset \bigcap_{x \in X \setminus S} X \setminus \{x\}$. Conversely, if z is a point in $X \setminus \{x\}$ for every x not in S, then z must be in S.

□