Regular polygons

Regular polygons with three sides (equilateral triangle), four sides (square), and six sides are easy to construct.

¶ **1** (Angle bisection)**.** It will be useful to know how to bisect an angle. If $\angle AOB$ is the given angle, we construct the circle with center O passing through A. This circle intersects $\overrightarrow{OA}$ in A and $\overrightarrow{OB}$ in D. The perpendicular bisector of segment $\overline{AD}$ passes through O and bisects $\angle AOB$.

¶ **2** (Square)**.** To construct a square inscribed in a circle, draw two perpendicular diameters to the given circle. These diameters will intersect the circle at the vertices of the square.

¶ **3** (Hexagon)**.** To construct a regular hexagon inscribed in a circle O^B: (a) draw the circle B^O, (b) let A and C be the points of intersection of O^B and B^O (c) with the point of the compass at C, draw the circle C^O, obtaining a new point D (d) with the point of the compass at D, draw the circle D^O, obtaining E, and (e) with the point at E, draw the circle E^O, obtaining a new point F.
The points A, B, C, D, E, and F are the vertices of the required regular hexagon

¶ **4** (Equilateral triangle)**.** To construct an equilateral triangle inscribed in circle, first construct and inscribed regular hexagon with vertices A, B, C, D, E, and F. Then A, C and E (for example) are the vertices of an inscribed equilateral triangle.

The Pentagon There are several methods of constructing a regular pentagon with ruler and compass.

¶ **5.** Given a circle with center O through A, follow these steps to construct a regular pentagon inscribed in O^A.

(a) Let $\overline{AB}$ be the diameter of O^A through A.

(b) Let E be the midpoint of $\overline{OA}$.

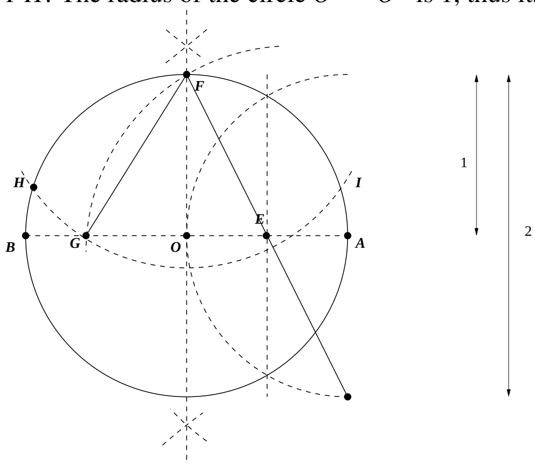
(c) Let F be a point of intersection of O^A and the perpendicular to $\overline{AB}$ at O. (This point F is the first vertex of the pentagon.)

(d) Let G be the intersection of $\overline{AB}$ and E^F.

(e) The circle F^G meets circle O^A at two new vertices H and I of the required pentagon.

(f) Draw two circles with centers H and I and with the same radius FG to find the last two vertices of the required pentagon.

¶ 6. The construction performed in Problem 5 produces the following figure. We will compute the length of segment $\overline{FH}$. The radius of the circle $O^A = O^B$ is 1, thus its diameter $AB = 2$.



(a) Use the Pythagorean Theorem on $\triangle OEF$ to prove that $FE = \dfrac{\sqrt{5}}{2}$.

(a) Prove that $OG = \dfrac{\sqrt{5}-1}{2}$.

(b) Use the Pythagorean Theorem on $\triangle GOF$ and the value of OG found in (b) to obtain $FG = \sqrt{\dfrac{5-\sqrt{5}}{2}}$.

¶ **7.** The next construction of a regular pentagon inscribed in a circle is somewhat similar, but it has the distinctive feature of using only a compass. This is mentioned for historical reasons, as it was a long standing problem to determine if every construction that could be performed with ruler and compass could also be performed with compass alone. This problem was eventually solved by the Italian geometer Mascheroni (1750–1800). The related problem, whether it is possible to perform all "ruler and compass" constructions could also be performed with ruler only, has a negative answer.
In order to construct an inscribed regular pentagon with compass alone, we take as given a circle, O^B, with center O and passing through B. The perform the following steps.

(a) Draw the circle, B^O, with center B passing through O.

(b) Let A and C be the points of intersection of B^O and O^B.

(c) Draw the circle C^O. This circle intersects O^B in B and in a new point D.

(d) Notice that A, B, C, and D are consecutive vertices of a regular hexagon inscribed in the circle O^B.

(e) Draw the circles with A and D as centers and AC as radius. Let X be one of the points of intersection.

(f) The line $\overleftrightarrow{OX}$ and the circle O^B meet at points F and K, with F being the midpoint of $\overline{BC}$.

(g) Draw circle F^O, meeting O^B at G and H.

(h) Let Y be the point at distance OX from both G and H and which is separated from X by O.

(i) The length of the segment AY is equal to a side of the required inscribed regular pentagon.

¶ **8.** It turns out that the next regular polygon in the list, namely, the heptagon, cannot be constructed with ruler and compass alone. This was discovered by Gauss (1777–1855) when he was just 19 year old. In fact, he completely determined precisely which regular n-sided polygons can be constructed with ruler and compass alone. To explain the content of Gauss Theorem, we need to recall prime number (a whole number > 1 which has no other factors but 1 and itself, and a peculiar family of prime numbers called Fermat primes. Fermat primes are prime numbers of the form $F_k = 2^{2^k} + 1$, where $k = 0, 1, 2, \cdots$ The first Fermat prime numbers are

$$\begin{aligned}
F_0 &= 2^{2^0} + 1 = 2^1 + 1 = 3 \\
F_1 &= 2^{2^1} + 1 = 2^2 + 1 = 5 \\
F_2 &= 2^{2^2} + 1 = 2^4 + 1 = 17 \\
F_3 &= 2^{2^3} + 1 = 2^8 + 1 = 257 \\
F_4 &= 2^{2^4} + 1 = 2^{16} + 1 = 65537
\end{aligned}$$

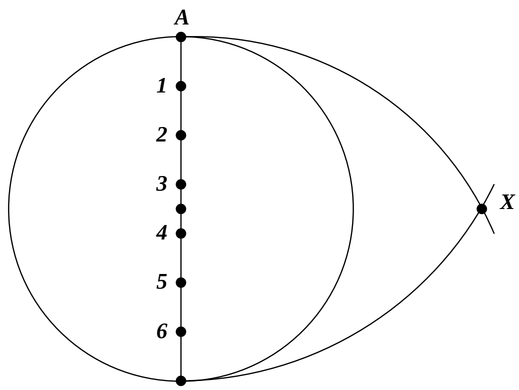
The next number in the list $F_5 = 2^{2^5} + 1 = 4294967297$ is not prime.

Theorem 1 (Gauss)**.** *A regular n-sided polygon can be constructed with ruler and compass alone if and only if all the odd prime factors of n are distinct Fermat primes.*

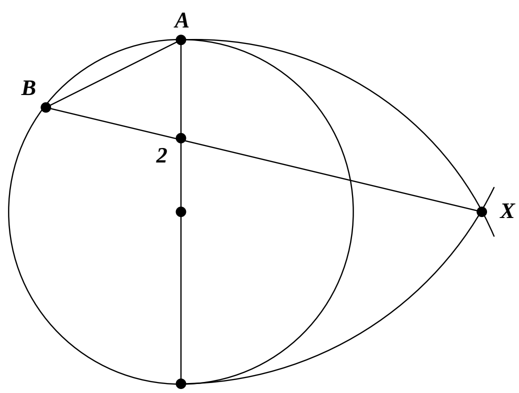
Use this theorem to list all the regular polygons with up to 100 sides which can be constructed with ruler and compass alone.

¶ **9.** I was taught in high-school the following construction of a regular polygon with n sides inscribed in a given circle. Here we consider the case of a heptagon, $n = 7$.

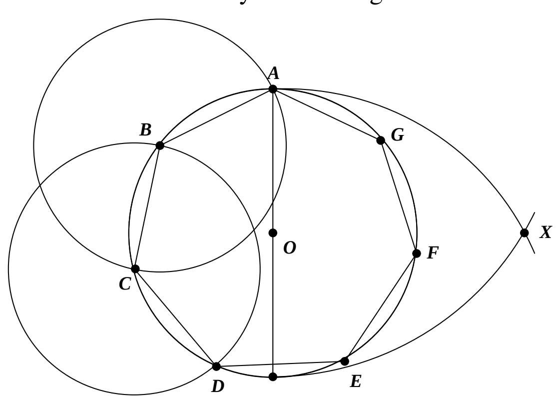
(a) Given circle O^A, draw circle from the points of intersection of O^A and $\overleftrightarrow{OA}$ to find point X. Then divide diameter through A into n equal parts (in this case, $n = 7$).



(b) Find the point B where the the ray from X and through the second point in the diameter meets the circle O^A. The point B is one vertex of the required polygon (in this case, heptagon) and the segment $\overline{AB}$ is one of the sides of the required polygon.



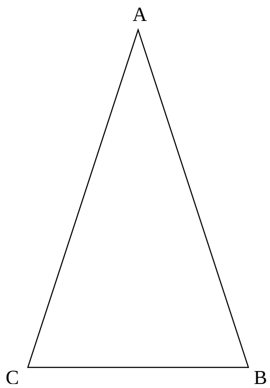
(c) Find the next vertex by intersecting the circle with center B and radius AB with the circle O^A, and so on.

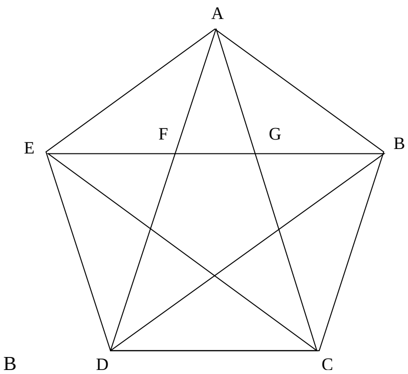


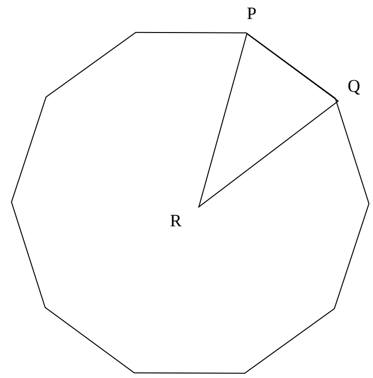
¶ **10** (The Golden Triangle)**.** A golden triangle is an isosceles triangle that has its two equal angles of double size the other angle. The figure on the left below depicts a golden triangle $\triangle ABC$. Angles are

$$\angle ABC = \angle ACB = 2\angle BAC = 72^\circ$$

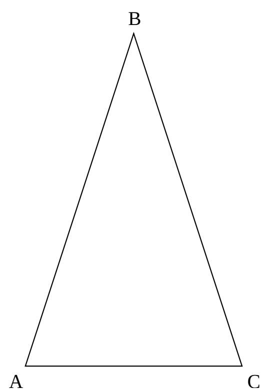
How many golden triangles can you find in the middle figure below?







¶ **11.** The golden triangle $\triangle ABC$ below has $\angle ABC = 36^\circ$.



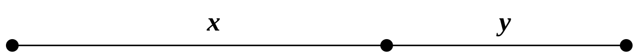
(a) Find D, the intersection of the ray bisector of $\angle BAC$ with segment $\overline{BC}$.

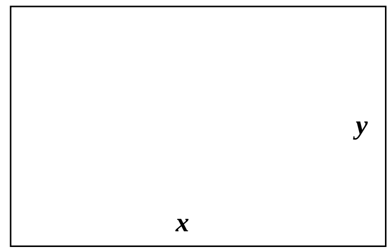
(b) Why is the proportion $\dfrac{AB}{AC} = \dfrac{AC}{CD}$ true?

Golden Ratio. Two numbers x and y, with $x > y$, are in golden ratio (or divine proportion) if

$$\frac{x+y}{x} = \frac{x}{y}$$

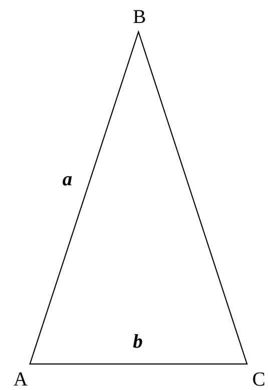
In words, the total length $x + y$ is to the largest length x as the largest length is to the smallest one.





¶ **12.** Use the quadratic formula to find the number $\phi > 1$ so that ϕ and 1 are in divine proportion.

¶ **13.** In the golden triangle $\triangle ABC$, let $a = AB = BC$ and $b = AC$.



(a) Use the proportion found in Problem 11, and some algebra, to show that $ab = (a + b)(a - b)$.

(b) Prove that $x = \dfrac{a^2 - b^2}{a}$ and $\dfrac{b^2}{a}$ are in golden ratio.

(c) If $a = AB = 1$, then find the value of b.

Construction of a Golden Triangle. We have proved that an isosceles triangle with long side equal to 1 and short side equal to $\frac{\sqrt{5}-1}{2}$ is a golden triangle. Problem 14 uses this information in order to construct a golden triangle.

¶ **14.** Given a segment $\overline{AB}$ of length a, follow these steps to construct a golden triangle with its longer sides congruent to AB.

(a) Draw a line $\overleftrightarrow{AB}$ through points A and B with $AB = a$.

(b) On the perpendicular to $\overleftrightarrow{AB}$ through A, find a point E such that $AB = AE$.

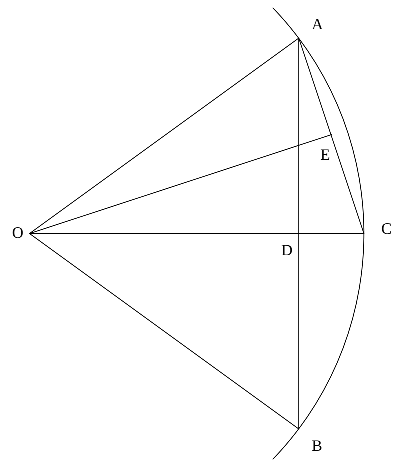
(c) From the middle point, M, of $\overline{AB}$ draw the circle through E, and let C be the point of intersection of M^E and $\overrightarrow{AB}$.

(d) Why is the ratio $\frac{BC}{AB} = \frac{\sqrt{5}-1}{2}$?

(e) Find the point D where the circle with center B and radius AB intersects the bisector of $\overline{BC}$.

(f) Why is $\triangle BDC$ a golden triangle?

The side length of a regular pentagon We will compute the length of the side of a regular pentagon inscribed in a circle of radius 1. In order to do that, we will compute first the length of the side of a regular decagon inscribed in the same circle.



¶ **15.** In the figure above, segment $\overline{AB}$ is one of the sides of a pentagon inscribed in a circle with center O and radius one, and segment $\overline{AC}$ is the side of a decagon inscribed in the same circle. Thus the lengths $OA = OC = OB = 1$. The angles $\angle AOB = 72°$ and $\angle AOC = 36°$.

(a) Why is the proportion $\dfrac{1}{AC} = \dfrac{AC}{2DC}$ true?

(b) Apply the Pythagorean Theorem to $\triangle ADC$ and simplify to obtain $AB = \sqrt{4AC^2 - AC^4}$.

(c) Because $\triangle OAC$ is a golden triangle and $OA = OC = 1$, the length $AC = \dfrac{\sqrt{5} - 1}{2}$. Use this value to obtain that AB, the length of the side of a regular pentagon inscribed in a circle of radius 1 is $AB = \sqrt{\dfrac{5 - \sqrt{5}}{2}}$.