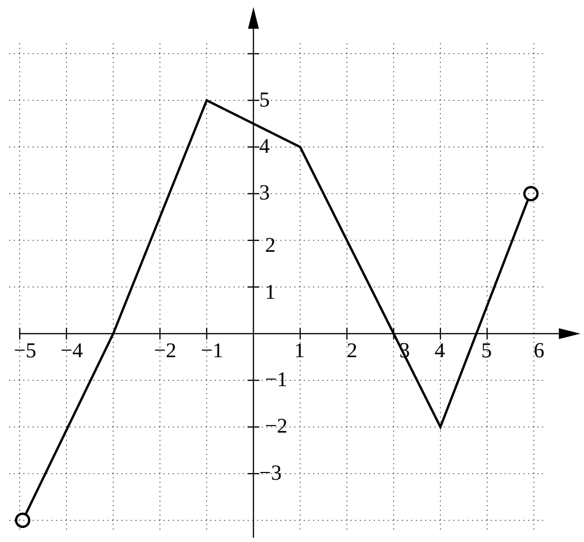
Math 102. 1st Midterm Solutions

Problem 1 The function f has the following graph.



(i) What are the values $f(-2)$, $f(-1)$, and $f(3)$? (ii) Determine the domain and the range of f. (Use interval notation.) (iii) Determine the intervals on which f is increasing and the intervals on which f is decreasing. (iv) Find the local maxima and the local minima of f. (v) Find the average rate of change of f between 3 and -2.

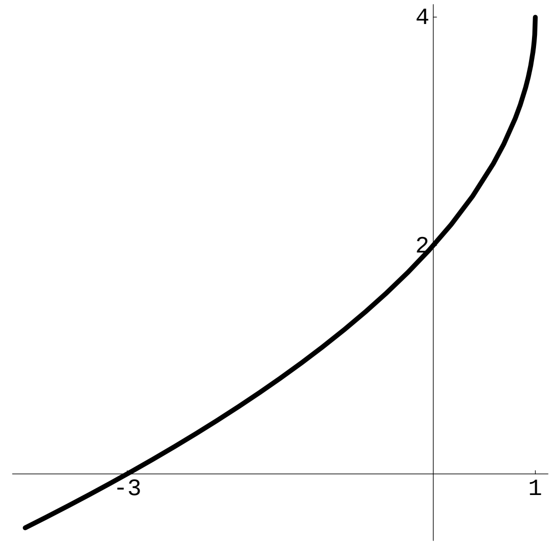
Solution. (i) $f(-2) = 2.5$, $f(-1) = 5$, $f(3) = 0$. (ii) Domain is $(-5, 6)$; Range is $(-4, 5]$. (iii) Increasing on $(-5, -1]$ and on $[4, 6)$; Decreasing on $[-1, 4]$. (iv) Local Max is 5; Local Min is -2. (v) Average rate of change is $-1/2$.

Problem 2 For the function $f(x) = 10x^2 - 7x + 3$ calculate the following values (simplify completely): (i) $f(a)$; (ii) $f(a + h)$; and (iii) $\dfrac{f(a + h) - f(a)}{h}$

Solution. (i) $\boxed{10a^2 - 7a + 3}$ (ii) $\boxed{10a^2 + 20ah + 10h^2 - 7a - 7h + 3}$ (iii) $\boxed{10h + 20a - 7}$

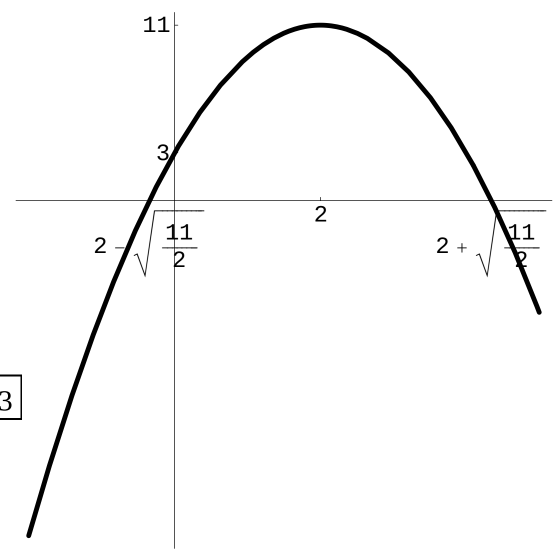
Problem 3 Use basic shapes and transformations to sketch the graph of $f(x) = 4 - 2\sqrt{1 - x}$. Label the intercepts, if any.

Solution. Do these steps: (1) start with the basic shape $y = \sqrt{x}$; (2) shift 1 unit left to graph $y = \sqrt{1 + x}$; (3) reflect about y-axis to graph $y = \sqrt{1 - x}$; (4) stretch vertically by 2 to graph $y = 2\sqrt{1 - x}$; (5) reflect about x-axis to graph $y = -2\sqrt{1 - x}$; and (6) shift up 4 units to graph $y = 4 - 2\sqrt{1 - x}$. Final graph is:



Problem 4 For the quadratic function $f(x) = -2x^2 + 8x + 3$: (i) Express it in standard form $f(x) = a(x - h)^2 + k$; (ii) Find its vertex and intercepts; (iii) Sketch its graph; and (iv) What is its maximum value?

Solution. (i) Standard form is $\boxed{f(x) = -2(x - 2)^2 + 11}$. (ii) Vertex is $\boxed{(2, 11)}$; x-intercepts are $\boxed{2 \pm \sqrt{\frac{11}{2}}}$; y-intercept is $\boxed{3}$. (iv) Max value is $\boxed{2}$. (iii) Graph:



Problem 5 A rancher has 264 feet of fencing to enclose to equal adjacent corrals. (i) What dimensions (for each corral) will make the area enclosed the largest? (ii) What is the maximum total area?

Solution. From the figure below we see that if x is the length and y is the width of one corral, then $3x + 4y = 264$. The total area is $A = (x)(2y) = 2xy$. Solving for y in $3x + 4y = 264$

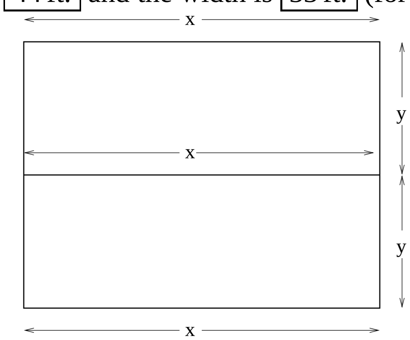
we obtain $y = 66 - \frac{3}{4}x$. Substituting this value in the expression for the area we obtain $A = 2x\left(66 - \frac{3}{4}x\right)$, and after simplifying we have

$$A = -\frac{3}{2}x^2 + 132x$$

This is a quadratic function in x whose graph is a parabola which opens down. We complete squares and write it in standard form:

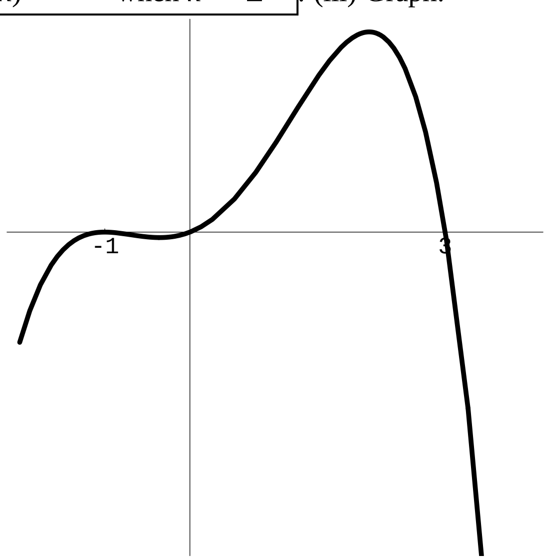
$$A = -\frac{3}{2}(x - 44)^2 + 2904$$

We see that the vertex is $(44, 2904)$, meaning that the maximum total area is $\boxed{2904\,\text{ft}^2}$, and is attained when the length is $\boxed{44\,\text{ft.}}$ and the width is $\boxed{33\,\text{ft.}}$ (for each corral).

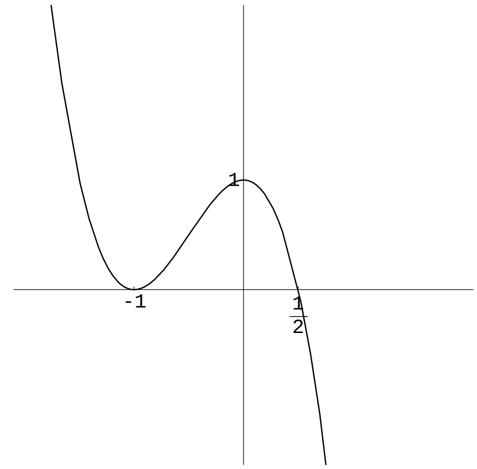


Problem 6 For the polynomial $P(x) = -2x(x+1)^2(x-3)$: (i) Find x and y-intercepts; (ii) Determine its end behavior; and (iii) Sketch its graph. (Make sure that your graph shows all intercepts and exhibits the proper end behavior.)

Solution. (i) x-intercepts are $\boxed{-1, 0, 3}$; y-intercept is $\boxed{0}$. (ii) $\boxed{P(x) \to -\infty \text{ when } x \to \pm\infty}$. (iii) Graph:



Problem 7 The following curve is the graph of a polynomial of degree 3. Find the polynomial.



Solution. The graph shows that the polynomial, $P(x)$, has a root of even multiplicity at -1 and a root of odd multiplicity at $1/2$. Since the total degree is 3, it must be $P(x) = a(x+1)^2(x - 1/2)$. Moreover, its y-intercept is 1, so $a = -2$. That is, $\boxed{P(x) = 2(x+1)^2(x - 1/2)}$.

Problem 8 For the rational function $R(x) = \dfrac{2 - 3x}{1 - x}$: (i) Find its intercepts; (ii) Find its vertical and its horizontal asymptotes; and (iii) Sketch its graph (make sure to include intercepts and asymptotes in your sketch).

Solution. (i) x-intercept is $\boxed{2/3}$; y-intercept is $\boxed{2}$. (ii) Vertical asymptote is $\boxed{x = 1}$; Horizontal asymptote is $\boxed{y = 3}$. (iii) Graph: